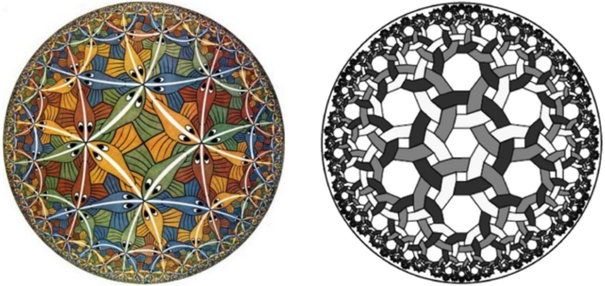
Understanding Holographic Entanglement

With gravitational path integral & tensor network

Bryan



Dunham, 2012 [1] on Escher's art

Introduction

- **Question:** how can we *"prepare"* / *construct a state*, e.g.
  - vacuum: $\rho = |0\rangle\langle 0|$
  - thermal: $\rho \propto \sum_n e^{-\beta H} |n\rangle\langle n|$

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- **Answer:** via holography! (*duh...*)
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Review: boundary entanglement as gravitational saddles

Recall: Path Integral in the Boundary & the Bulk

Lewkowycz & Maldacena, 1304.4926 [3]: "Generalized gravitational entropy"

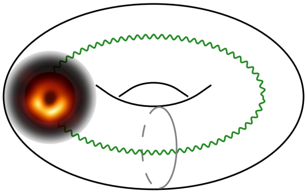

Figure: Thermal $\mathcal{Z}_\beta$

Thermal partition function
in the bulk & the boundary

Image made from *Benjamin, Collier & Maloney*, 2004.14428 [2] and the EHT black hole photo

- In *any* field theory, a thermal state can be prepared by a path integral;
- In a *holographic* theory, $\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk}$, a boundary state can be prepared by a bulk path integral.
  - e.g. the thermo-field double $\leftrightarrow$ the BTZ black hole Note that the Euclidean BTZ geometry is smooth: filling in the t_E cycle (*not* the ϕ cycle) of the torus
  - c.f. Chern-Simons/WZW: not quite the same

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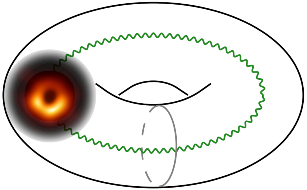

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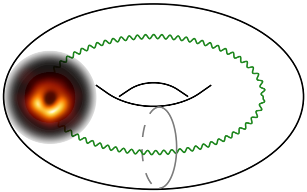

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$$\mathcal{Z}_{\partial B} = \mathcal{Z}_{Bulk} \tag{1}$$

- Replica trick: entanglement entropy for a region R:

$$S_R = \operatorname{Tr} \rho_R \log \rho_R \;\Leftarrow\; \operatorname{Tr} \rho_R^n \equiv \mathcal{Z}_n \tag{2}$$

i.e. reduced to the partition function of the n-replica.
It can be deployed in the boundary & the bulk!

  - Boundary: static geometry, but the field theory is usually strongly coupled — often difficult!
  - Bulk: dynamic geometry, weakly coupled gravity: gravity fills in the bulk smoothly, $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$: sum over classical saddles

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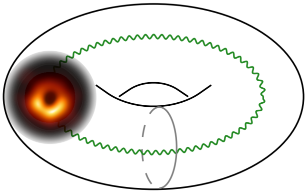

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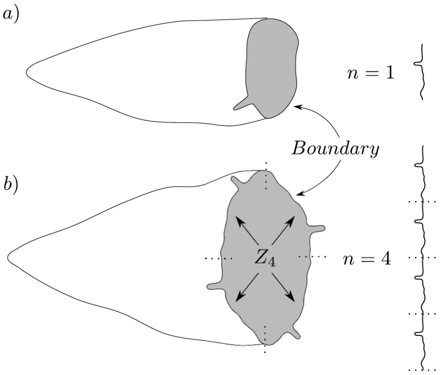

Figure: The bulk replica [3]

- $\mathcal{Z} \sim \sum_i e^{-S_i[g_{\mu\nu}]}$, saddle pt. approx.
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- $\widetilde{\mathcal{M}}_n/\mathbb{Z}_n$: conical singularity at the $\mathbb{Z}_n$; $n \to 1$,
  $\Rightarrow$ minimize the area of the $\mathbb{Z}_n$ fixed point
  $\Rightarrow$ the extremal surface, the RT surface
- **Lesson:** use bulk path integral to:
  - prepare the states
  - compute the entanglement entropy

This requires holography,
but not necessarily $\mathrm{AdS/CFT}$!

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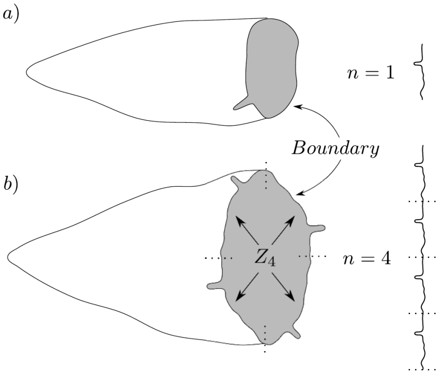

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Compare: bulk emergence from tensor networks

Prepare states via Tensor Networks

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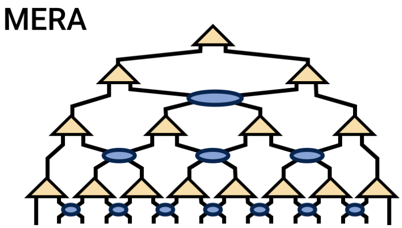

Figure: MERA

Multi-scale Entanglement Renormalization Ansatz

Image from tensornetwork.org

- Gravitational path integral: a spacetime perspective
  Tensor network: on a constant time slice
- States constructed with tensor networks: common in condensed matter (e.g. DMRG)
- To find the ground state of a system
  - Write down a tensor network as an **ansatz** for the ground state;
  - Vary the components of each tensor to achieve minimal energy — **optimization!**

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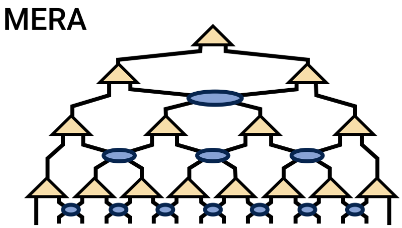

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AdS Bulk as a Tensor Network

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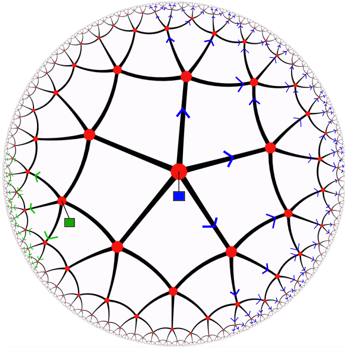

Figure: The HaPPY code [7, 8]

- Gravitational path integral: a spacetime perspective
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- Dual geometry in the bulk is understood as the “continuous limit” of a *tensor network*
  - Node: tensor acting on the Hilbert space
  - Leg: index to be contracted
    - $1 \times$ free leg
      takes in bulk local operator insertions
    - $5 \times$ contracted leg
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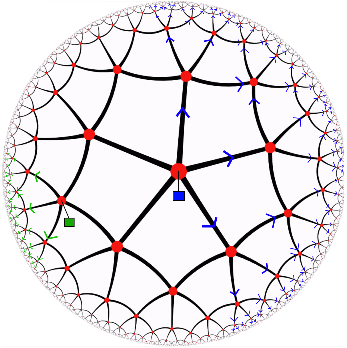

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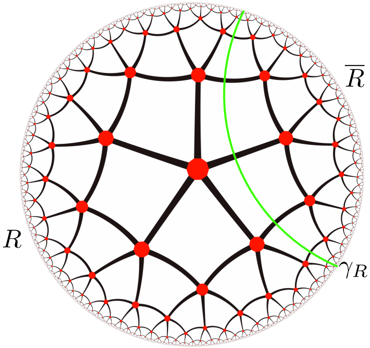

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- "Entanglement" between R and $\bar{R}$:
  $\propto$ min # of links connecting the two regions
  Naturally, entropy $=$ bulk area!
- "Complexity": # of nodes
  Proposal: complexity $=$ bulk volume!
  See e.g. *Susskind*, 1403.5695 [9]
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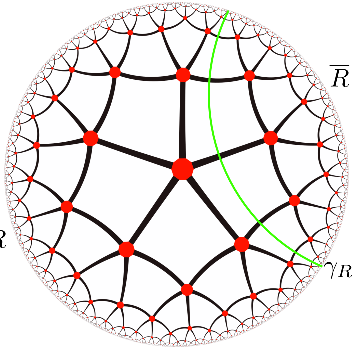

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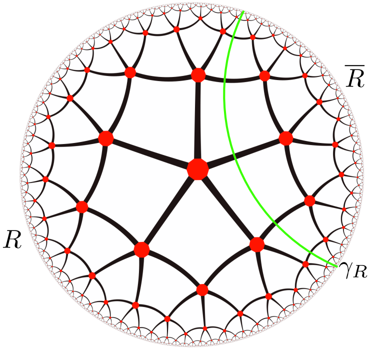

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The Lesson

- The Ryu–Takayanagi proposal: $S \sim \frac{A}{4G_N}$
  - ... seems to be universal in holographic systems,
  - ... where boundary states can be constructed from some sort of bulk operations:
    - Gravitational path integral
    - Tensor network
- Applications: beyond standard $\mathrm{AdS}_3/\mathrm{CFT}_2$
  - Cutoff holography: *Lewkowycz, Liu, Silverstein & Torroba*, 1909.13808 [12]
  - Flat holography: *Apolo, Jiang, Song & Zhong*, 2006.10740 [13, 14]

Application: cutoff AdS_3 / $T\bar{T}$ deformed theory

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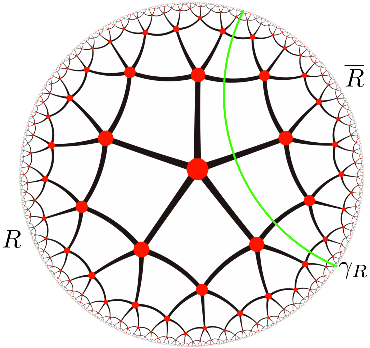

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- AdS_3 with finite cutoff:
  holographic renormalization of the boundary theory
  - This is clear in the tensor network picture "coarse-graining"
- Deform the boundary CFT_2 with some operator: $\mathrm{CFT}_2^{(\mathrm{UV})} \rightsquigarrow$ deformed theory $^{(\mathrm{IR})}$
- Surprisingly, we were able to find the deformed theory! $\delta S \propto \mu\,(T\bar{T})_\mu,\ T\bar{T} = \frac{1}{8}\left(T^{\alpha\beta}T_{\alpha\beta} - (T^\alpha_\alpha)^2\right)$
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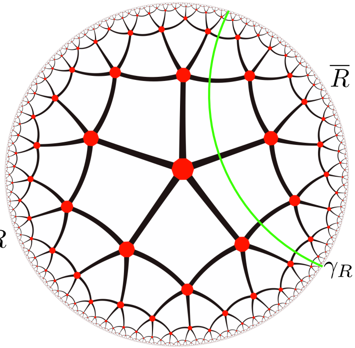

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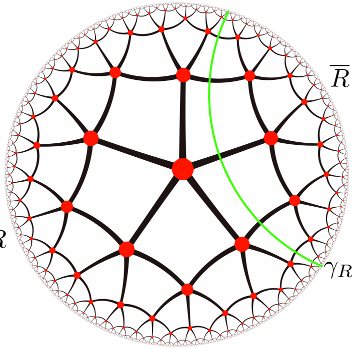

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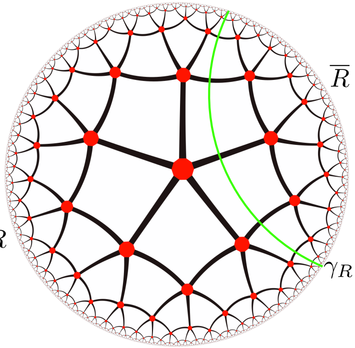

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  - … *analytically continued* from $\mathbb{Z}_n$ to $U(1)$,
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  - This would in turn give us a hint of the modular flow in the $T\bar{T}$ deformed theory! (*ongoing work*)

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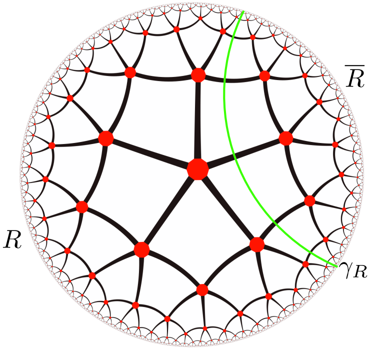

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Further Reading & Outlook

- Single-trace $T\bar{T}$ duality and the flow towards to UV
  - Luis Apolo, Stephane Detournay & Wei Song. *TsT, $T\bar{T}$ and black strings*. *JHEP*. **06**:109, **2020**. arXiv: 1911.12359 [hep-th]
- Quantum error correction:
  - Alexander Jahn & Jens Eisert. *Holographic tensor network models and quantum error correction: A topical review*. **February 2021**. arXiv: 2102.02619 [quant-ph]
- Tensor network for flat spacetime?
  - Alex May. *Tensor networks for dynamic spacetimes*. *JHEP*. **06**:118, **2017**. arXiv: 1611.06220 [hep-th]

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