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$T\bar{T}$ -deformed CFTs with $\mu < 0$ have been proposed by McGough, Mezei & Verlinde, [1611.03470](https://arxiv.org/abs/1611.03470) [1] to be holographically dual to Einstein gravity with a finite Dirichlet cutoff.

We generalize the proposal for $\mu > 0$ by *gluing-on* another patch of AdS_3 , and provide various evidence for this extended holography, now valid for $\mu \in \mathbb{R}$.

Pure Einstein gravity without matter in AdS_3 :

$$ds^2 = \ell^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{(du + \rho \bar{\mathcal{L}}(v) dv)(dv + \rho \mathcal{L}(u) du)}{\rho} \right) = n_\mu n_\nu dx^\mu dx^\nu + \frac{1}{\zeta} \gamma_{ij} dx^i dx^j, \quad (1)$$

- Transverse coordinates: $x^i \sim \varphi, t$
Light-cone coordinates: $u, v = \varphi \pm t$
 $\mathcal{L}(u), \bar{\mathcal{L}}(v)$: arbitrary periodic functions
- Radial coordinate: $\ell^{-2} g_{\varphi\varphi} \equiv r^2 \equiv \zeta^{-1}$
 r : the “proper radius”

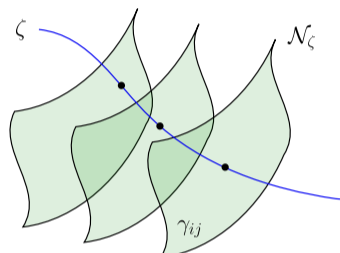


Image courtesy: R. Szalai, DOI:10.1007/s11071-020-05891-1

- Structure: foliated by constant ζ surfaces \mathcal{N}_ζ
Asymptotic boundary: \mathcal{N}_0 is at $\zeta \rightarrow 0$
Fefferman & Graham, [0710.0919](https://arxiv.org/abs/0710.0919) [2]
Banados, Teitelboim & Zanelli, [hep-th/9204099](https://arxiv.org/abs/hep-th/9204099) [3]

“ $T\bar{T}$ ” deformation as the flow of action:

$$\partial_\mu I = 8\pi \int d^2x T\bar{T}(\mu) = \pi \int d^2x (T^{ij} T_{ij} - (T^i_i)^2)_{(\mu)}, \quad x, \bar{x} = \varphi' \pm t' \quad (2)$$

- **Irrelevant:** initially CFT_2 : $T\bar{T}(\mu=0) = T_{xx} T_{\bar{x}\bar{x}}$, but conformal symmetry will be broken for $\mu \neq 0$.
- **Solvable:** the deformed spectrum of $\hat{H}(\mu)$ and $\hat{J}(\mu)$ on a cylinder of radius R is a simple function:

$$E(\mu) = -\frac{R}{2\mu} \left(1 - \sqrt{1 + \frac{4\mu}{R} E(0) + \frac{4\mu^2}{R^4} J(0)^2} \right), \quad (3)$$

$$J(\mu) = J(0)$$

of the undeformed spectrum $E(0), J(0)$.
(under reasonable assumptions)

Zamolodchikov, [hep-th/0401146](https://arxiv.org/abs/hep-th/0401146) [4]
Dubovsky, Flauger & Gorbenko, [1205.6805](https://arxiv.org/abs/1205.6805) [5]
Dubovsky, Gorbenko & Mirbabayi, [1305.6939](https://arxiv.org/abs/1305.6939) [6]
Smirnov & Zamolodchikov, [1608.05499](https://arxiv.org/abs/1608.05499) [7]
Cavaglià, Negro, Szécsényi & Tateo, [1608.05534](https://arxiv.org/abs/1608.05534) [8] et al

Cutoff AdS_3 duality:

Holography within a finite Dirichlet wall

McGough, Mezei & Verlinde, [1611.03470](https://arxiv.org/abs/1611.03470) [1]
Kraus, Liu & Marolf, [1801.02714](https://arxiv.org/abs/1801.02714) [9] et al

Dictionary: radial location ζ_c of the cutoff surface \mathcal{N}_{ζ_c} gets mapped to the deformation parameter μ :

$$\zeta_c = -\frac{c\mu}{3\ell^2} \quad (4)$$

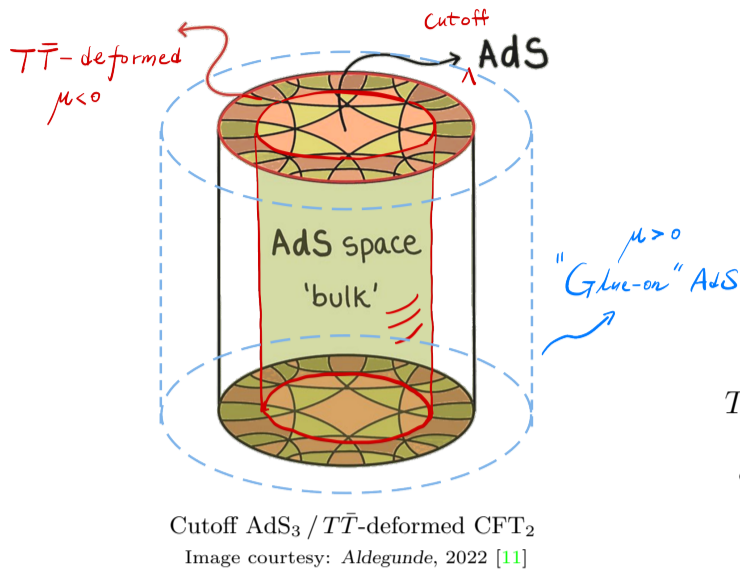
$T\bar{T}$ flow recast geometrically as the i, j components of the Einstein equations.

This is a “non-AdS” / non-CFT duality.
A step towards quantum gravity in realityTM!

Caveat: the duality only admits $\zeta_c > 0$ so $\mu < 0$.
But $T\bar{T}$ itself admits $\mu > 0$ with nice properties.

The related proposal of Guica & Monten, [1906.11251](https://arxiv.org/abs/1906.11251) [10] admits both signs of μ .

What is the other side of the duality?

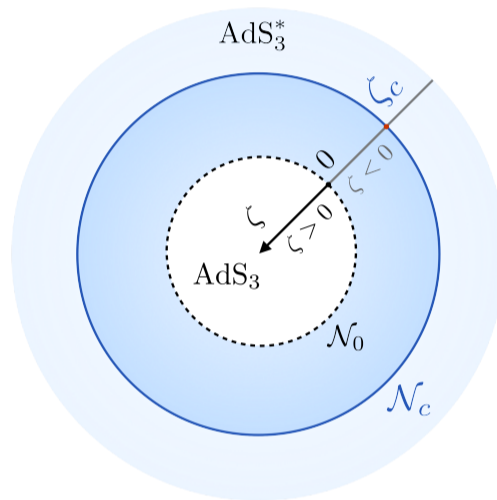


Cutoff AdS_3 / $T\bar{T}$ -deformed CFT_2
Image courtesy: Aldegunde, 2022 [11]

Proposal:

Glue-on AdS_3 – beyond the infinity

Cutoff / glue-on AdS_3 Gravity $\equiv T\bar{T}$ deformed CFT_2 at \mathcal{N}_{ζ_c}
 $\zeta_c > 0$ / $\zeta_c < 0$ $\mu \in \mathbb{R}$



Top-down view of a constant t slice

- The metric (1) has a simple pole, thus admits a well-defined **analytic continuation**:

$$\frac{1}{\zeta} \equiv \ell^{-2} g_{\varphi\varphi} \in \mathbb{R}, \quad \zeta_c = -\frac{c\mu}{3\ell^2} \in \mathbb{R} \quad (5)$$

What are we doing other than copy-pasting?
“Renormalize” the divergences as $\zeta \rightarrow 0^\pm$:

- introduce $\mathcal{N}_{\zeta=\pm\epsilon}$ and “glue” them together;
- i.e. exclude the $-\epsilon < \zeta < \epsilon$ region, until finally sending $\epsilon \rightarrow 0$.

- **Extending the flow:** matching energy momentum (Brown-York) & the $T\bar{T}$ flow equations:

$$T_{ij} = \frac{\sigma_{\zeta_c}}{8\pi G} \left(K_{ij} - K\gamma_{ij} + \frac{1}{\ell|\zeta_c|} \gamma_{ij} \right), \quad (6)$$

$$\sigma_{\zeta_c} = \frac{|\zeta_c|}{\zeta_c} = \pm 1, \quad \gamma_{ij} = \zeta_c h_{ij},$$

h_{ij} : the induced metric.

The field theory metric γ_{ij} is always positive-definite, while h_{ij} becomes negative-definite for the glue-on region $\rho, \zeta < 0$.

This discrepancy is the origin of $|\zeta_c|$.

- Demanding a **non-singular extended geometry** reproduces bounds on the $T\bar{T}$ deformed theory, e.g.

$$\zeta_c \geq -1 \Leftrightarrow \mu \leq \frac{3\ell^2}{c} \quad (7)$$

This is a Killing horizon in the glue-on region of the extended geometry; $(\det g_{\mu\nu})_{\zeta=-1} = 0$.

- **Spectrum from the extended geometry:** they are conserved charges \mathcal{Q} of the Killing symmetries, and can be computed with the *covariant formalism*.

Iyer & Wald, Barnich & Brandt et al
With $T\bar{T}$: Kraus, Monten & Myers, [2103.13398](https://arxiv.org/abs/2103.13398) [14]

$$\delta E(\mu) \equiv \ell^{-1} \delta \mathcal{Q}_{\partial_t}, \quad \delta J(\mu) \equiv \delta \mathcal{Q}_{\partial_\varphi}.$$

This reproduces $E(\mu), J(\mu)$ with $\mu \in \mathbb{R}$ in (3).

(t', φ') : normalized boundary coordinates:

$$ds_c^2 = \ell^2 (-dt'^2 + d\varphi'^2), \quad \varphi' \sim \varphi' + 2\pi. \quad (8)$$

- **State-dependent maps of coordinates:**

$$dt' = \sqrt{(1 - \zeta_c(T_u + T_v)^2)(1 - \zeta_c(T_u - T_v)^2)} dt, \quad d\varphi' = d\varphi + \zeta_c(T_u^2 - T_v^2) dt. \quad (9)$$

\rightsquigarrow correct charges, the modified signal propagation speed $v'_\pm \equiv \pm d\varphi'/dt'$, and $T\bar{T}$ thermodynamics upon Wick rotation. In particular,

$$\mu > 0, \quad T_L(\mu) T_R(\mu) \leq -\frac{1}{4\pi^2 \ell^2 \zeta_c} = \frac{3}{4\pi^2 c\mu} = T_H(\mu)^2.$$

$T_{L,R}$: temperatures associated with $u', v' = \varphi' \pm t'$.
 T_H : the Hagedorn temperature: exceeding T_H leads to a complex entropy.

Giveon, Itzhaki & Kutasov, [1701.05576](https://arxiv.org/abs/1701.05576) [15]
Apolo, Detournay & Song, [1911.12359](https://arxiv.org/abs/1911.12359) [16]

$T\bar{T}$ partition functions from the bulk

- Via bulk on-shell action of the dominant saddle:

$$Z_{T\bar{T}}(\mu) = \mathcal{Z}(\zeta_c) \approx e^{-I(\zeta_c)}. \quad (10)$$

$I(\zeta_c)$ diverges at $\zeta \rightarrow 0^\pm$: we need *holographic renormalization*, with the boundary action:

$$I_{\mathcal{N}_\zeta} := -\frac{1}{8\pi G} \int_{\mathcal{N}_\zeta} d^2x \sqrt{h} h^{ij} K_{ij} \quad (11)$$

$$+ \frac{\sigma_\zeta}{8\pi G} \int_{\mathcal{N}_\zeta} d^2x \sqrt{h} \left(\ell^{-1} - \frac{\ell \mathcal{R}[h]}{4} \log |\zeta| \right).$$

Henningson & Skenderis, [hep-th/9806087](https://arxiv.org/abs/hep-th/9806087) [17] et al

$\log |\zeta|$: not diff-invariant, due to the Weyl anomaly.
 $I_{\mathcal{N}_\zeta}$: consistent with the B-Y stress tensor (6).

$Z_{T\bar{T}}$ agrees with the field theory analysis, using (9):

- **Torus:** modular invariance & sparseness of the “light” spectrum at large $c \rightsquigarrow$ universal form:

$$\log Z_{T\bar{T}}(\mu) \approx \begin{cases} -\frac{1}{2} (\beta_L + \beta_R) RE_{\text{vac}}(\mu), & \beta_L \beta_R > 1, \\ -2\pi^2 \left(\frac{1}{\beta_L} + \frac{1}{\beta_R} \right) RE_{\text{vac}} \left(\frac{4\pi^2}{\beta_L \beta_R} \mu \right), & \beta_L \beta_R < 1. \end{cases}$$

Datta & Jiang, [1806.07426](https://arxiv.org/abs/1806.07426) [18]

Apolo, Song & Yu, [2301.04153](https://arxiv.org/abs/2301.04153) [19]

cf. Hartman, Keller & Stoica, [1405.5137](https://arxiv.org/abs/1405.5137) [20]

- **Sphere:** maximally symmetric,

Donnelly & Shyam, [1806.07444](https://arxiv.org/abs/1806.07444) [21]

$$\langle T_{ij} \rangle = -\frac{1}{4\pi\mu} \left(1 - \sqrt{1 - \frac{c\mu}{3R^2}} \right) \gamma_{ij}. \quad (12)$$

Trace relation: $(-R) \partial_R \log Z_{T\bar{T}}(\mu) = \int d^2x \sqrt{\gamma} \langle T^i_i \rangle$
and the flow equation (2) admit the general solution with a μ -independent integration constant a :

$$\log Z_{T\bar{T}}(\mu, a) = \frac{\epsilon}{3} \log \left[\frac{R}{a} \left(1 + \sqrt{1 - \frac{c\mu}{3R^2}} \right) \right] - \frac{R^2}{\mu} \sqrt{1 - \frac{c\mu}{3R^2}} + \frac{R^2}{\mu}.$$

- $a = \sqrt{c|\mu|/3}$: recover Donnelly & Shyam [21]
- $a = \epsilon$: also a valid choice, where the RG length scale ϵ is decoupled from μ .

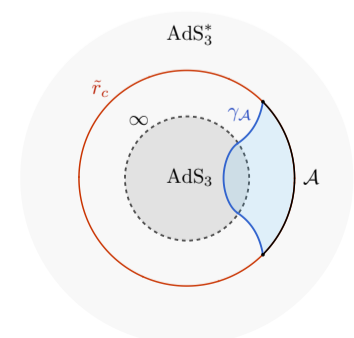
- Enlarge the **space of $T\bar{T}$ deformed theories**: with independent parameters (μ, a) .

- The $\log |\zeta|$ in (11) guarantees that $I = -\log Z_{T\bar{T}}$ satisfies the $T\bar{T}$ flow (2); not the case for [21].

Caputa, Datta, Jiang & Kraus, [2011.04664](https://arxiv.org/abs/2011.04664) [22]
Li, [2012.14414](https://arxiv.org/abs/2012.14414) [23]

- Future: understand the **entanglement structure** of $T\bar{T}$ deformation with the help of bulk geometry.

Lewkowycz, Liu, Silverstein & Torroba, [1909.13808](https://arxiv.org/abs/1909.13808) [24]



RT surface for the extended AdS_3

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