

Abstract: $T\bar{T}$ -deformed CFTs with $\mu < 0$ have been proposed by McGough, Mezei & Verlinde, [1611.03470](https://arxiv.org/abs/1611.03470) [1] to be holographically dual to Einstein gravity with a finite Dirichlet cutoff.

In this work we put forward a holographic proposal for $T\bar{T}$ -deformed CFTs with $\mu > 0$, in which case the bulk dual geometry is constructed by gluing-on a patch of AdS_3 to the original spacetime. We provide various evidence for this extended holography, now valid for $\mu \in \mathbb{R}$.

We start from **pure Einstein gravity** without matter in AdS_3 ; the Einstein equations are solved by the metric:

$$ds^2 = \ell^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{(du + \rho \bar{\mathcal{L}}(v) dv)(dv + \rho \mathcal{L}(u) du)}{\rho} \right) \quad (1)$$

$$= n_\mu n_\nu dx^\mu dx^\nu + \frac{1}{\zeta} \gamma_{ij} dx^i dx^j, \quad \frac{1}{\zeta} \equiv \ell^{-2} g_{\varphi\varphi} \equiv r^2$$

The radial coordinate: $\rho \approx r^{-2} \equiv \zeta$, where r is the “proper radius”; $\mathcal{L}(u)$ and $\bar{\mathcal{L}}(v)$ are arbitrary periodic functions.

The geometry is foliated by constant ζ surfaces \mathcal{N}_ζ . The asymptotic boundary \mathcal{N}_0 is at $\zeta \rightarrow 0$, with coordinates: $x^i \sim \varphi, t$, while $u, v = \varphi \pm t$.

Fefferman & Graham, [0710.0919](https://arxiv.org/abs/0710.0919) [2]

Banados, Teitelboim & Zanelli, [hep-th/9204099](https://arxiv.org/abs/hep-th/9204099) [3]

Banados, [hep-th/9901148](https://arxiv.org/abs/hep-th/9901148) [4]

Foundations: $\text{AdS}_3/\text{CFT}_2$ holography

asympt. AdS_3 Gravity \equiv Large N CFT_2 at \mathcal{N}_0

Maldacena, [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200) [5]

Aharony, Gubser, Maldacena, Ooguri & Oz, [hep-th/9905111](https://arxiv.org/abs/hep-th/9905111) [6] *et al*

- Define “ $T\bar{T}$ ” deformation as the flow of action:

$$\partial_\mu I = 8\pi \int d^2x T\bar{T}_{(\mu)} = \pi \int d^2x (T^{ij}T_{ij} - (T^i_i)^2)_{(\mu)} \quad (2)$$

where $x, \bar{x} = \varphi' \pm t'$. We shall start from CFT_2 such that $T\bar{T}_{(\mu=0)} = T_{xx}T_{\bar{x}\bar{x}}$, but note that the deformation is *irrelevant*: conformal symmetry will be broken for $\mu \neq 0$.

- Solvable:** the deformed spectrum of $\hat{H}(\mu)$ and $\hat{J}(\mu)$ on a cylinder of radius R is a simple function:

$$E(\mu) = -\frac{R}{2\mu} \left(1 - \sqrt{1 + \frac{4\mu}{R} E(0) + \frac{4\mu^2}{R^4} J(0)^2} \right), \quad (3)$$

$$J(\mu) = J(0)$$

of the undeformed spectrum $E(0), J(0)$, under reasonable assumptions (e.g. translation invariance).

Zamolodchikov, [hep-th/0401146](https://arxiv.org/abs/hep-th/0401146) [7]

Dubovsky, Flauger & Gorbenko, [1205.6805](https://arxiv.org/abs/1205.6805) [8]

Dubovsky, Gorbenko & Mirbabayi, [1305.6939](https://arxiv.org/abs/1305.6939) [9]

Smirnov & Zamolodchikov, [1608.05499](https://arxiv.org/abs/1608.05499) [10]

Cavaglià, Negro, Szécsényi & Tateo, [1608.05534](https://arxiv.org/abs/1608.05534) [11]

Dubovsky, Gorbenko & Mirbabayi, [1706.06604](https://arxiv.org/abs/1706.06604) [12] *et al*

Cutoff AdS_3 duality:

Holography within a finite Dirichlet wall

McGough, Mezei & Verlinde, [1611.03470](https://arxiv.org/abs/1611.03470) [1]

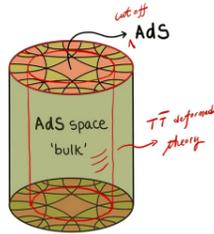
Kraus, Liu & Marolf, [1801.02714](https://arxiv.org/abs/1801.02714) [13] *et al*

Dictionary: the $T\bar{T}$ deformed theory lives on \mathcal{N}_{ζ_c} with metric γ_{ij} . Location (radius) of the cutoff surface gets mapped to the deformation parameter μ :

$$\zeta_c = -\frac{c\mu}{3\ell^2} \quad (4)$$

$T\bar{T}$ flow is recast geometrically as the i, j components of the Einstein equations. This is a “non- AdS ” / non-CFT duality.

A step towards quantum gravity in realityTM!



Cutoff $\text{AdS}_3 / T\bar{T}$ -deformed CFT_2

Image courtesy: Aldegunde, 2022 [14]

Caveat: the duality only admits $\zeta_c > 0$ so $\mu < 0$. But $T\bar{T}$ itself admits $\mu > 0$ with cool properties.

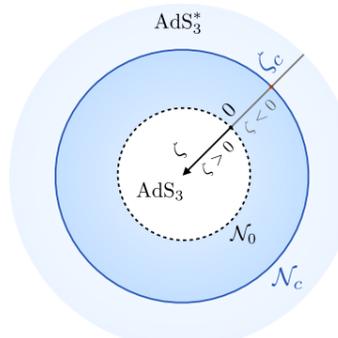
The related proposal of Guica & Monten, [1906.11251](https://arxiv.org/abs/1906.11251) [15] admits both signs of μ .

What is the other side of the duality?

Proposal:

Glue-on AdS_3 – beyond the infinity

Cutoff / glue-on AdS_{d+1} Gravity $\equiv T\bar{T}$ deformed CFT_d at \mathcal{N}_{ζ_c}



Top-down view of a constant t slice

- Analytic continuation:** metric (1) has a simple pole, thus admits well-defined analytic continuation:

$$\frac{1}{\zeta} \equiv \ell^{-2} g_{\varphi\varphi} \in \mathbb{R}, \quad \zeta_c = -\frac{c\mu}{3\ell^2} \in \mathbb{R} \quad (5)$$

What are we doing other than copy-pasting?

- Physical quantities diverge as $\zeta \rightarrow 0$. A prescription is required to “renormalize” the divergences.
- Treatment for the singularity at $\zeta \rightarrow 0^\pm$: introduce $\mathcal{N}_{\zeta=\pm\epsilon}$ and glue them together (“glue-on”); exclude the $-\epsilon < \zeta < \epsilon$ region until finally sending $\epsilon \rightarrow 0$.

- Extending the flow:** matching energy momentum (Brown-York) & the $T\bar{T}$ flow equations:

$$T_{ij} = \frac{\sigma_{\zeta_c}}{8\pi G} \left(K_{ij} - K\gamma_{ij} + \frac{1}{\ell|\zeta_c|} \gamma_{ij} \right), \quad (6)$$

$$\sigma_{\zeta_c} = \frac{|\zeta_c|}{\zeta_c} = \pm 1, \quad \gamma_{ij} = \zeta_c h_{ij},$$

where h_{ij} is the induced metric.

The field theory metric γ_{ij} is always positive-definite, while h_{ij} becomes negative-definite for the glue-on region $\rho, \zeta < 0$. This discrepancy is the origin of the $|\zeta_c|$.

- Demanding a **non-singular extended geometry** reproduces bounds on the $T\bar{T}$ deformed theory, e.g.

$$\zeta_c \geq -1 \Leftrightarrow \mu \leq \frac{3\ell^2}{c} \quad (7)$$

This is a Killing horizon in the glue-on region of the extended geometry; $(\det g_{\mu\nu})_{\zeta=-1} = 0$.

- Spectrum from the extended geometry:** they are conserved charges of the Killing symmetries, and can be computed with the *covariant formalism*.

Iyer & Wald, Barnich & Brandt *et al*

With $T\bar{T}$: Kraus, Monten & Myers, [2103.13398](https://arxiv.org/abs/2103.13398) [18]

$$\delta E(\mu) \equiv \ell^{-1} \delta \mathcal{Q}_{\partial_t}, \quad \delta J(\mu) \equiv \delta \mathcal{Q}_{\partial_{\varphi'}}.$$

This reproduces $E(\mu), J(\mu)$ with $\mu \in \mathbb{R}$ in (3).

$$ds_c^2 = \ell^2 (-dt'^2 + d\varphi'^2), \quad \varphi' \sim \varphi' + 2\pi. \quad (8)$$

This is realized by:

$$dt' = \sqrt{(1 - \zeta_c(T_u + T_v)^2)(1 - \zeta_c(T_u - T_v)^2)} dt,$$

$$d\varphi' = d\varphi + \zeta_c(T_u^2 - T_v^2) dt.$$

- The above **state-dependent map of coordinates** leads to the correct charges, the modified signal propagation speed $v'_\pm \equiv \pm d\varphi'/dt'$, and the $T\bar{T}$ thermodynamics when Wick rotated. In particular,

$$\mu > 0, \quad T_L(\mu) T_R(\mu) \leq -\frac{1}{4\pi^2 \ell^2 \zeta_c} = \frac{3}{4\pi^2 c\mu} = T_H(\mu)^2.$$

$T_{L,R}$ are temperatures associated with $u', v' = \varphi' \pm t'$.

T_H is the Hagedorn temperature: exceeding T_H leads to a complex entropy.

Giveon, Itzhaki & Kutasov, [1701.05576](https://arxiv.org/abs/1701.05576) [19]

Apolo, Detournay & Song, [1911.12359](https://arxiv.org/abs/1911.12359) [20]

$T\bar{T}$ partition functions from bulk actions

- The $T\bar{T}$ partition function is approximated by the bulk on-shell action of the dominant saddle:

$$Z_{T\bar{T}}(\mu) = \mathcal{Z}(\zeta_c) \approx e^{-I(\zeta_c)}. \quad (9)$$

$I(\zeta_c)$ diverges at $\zeta \rightarrow 0^\pm$. We apply a natural extension of *holographic renormalization* to remove the divergence. It includes the boundary action (with counterterms):

$$I_{\mathcal{N}_\zeta} := -\frac{1}{8\pi G} \int_{\mathcal{N}_\zeta} d^2x \sqrt{h} h^{ij} K_{ij}$$

$$+ \frac{\sigma_{\zeta_c}}{8\pi G} \int_{\mathcal{N}_\zeta} d^2x \sqrt{h} \left(\ell^{-1} - \frac{\ell \mathcal{R}[h]}{4} \log |\zeta| \right). \quad (10)$$

The lack of diff invariance of the $\log |\zeta|$ counterterm is known to be a reflection of the Weyl anomaly. The action is consistent with the Brown-York stress tensor in (6).

Henningson & Skenderis, [hep-th/9806087](https://arxiv.org/abs/hep-th/9806087) [21] *et al*

The resulting $I = -\log \mathcal{Z}$ satisfies the $T\bar{T}$ flow (2). Furthermore, it agrees with the field theory analysis:

- Torus:** modular invariance & sparseness of the “light” spectrum at large c leads to the universal form :

$$\log Z_{T\bar{T}}(\mu) \approx \begin{cases} -\frac{1}{2} (\beta_L + \beta_R) RE_{\text{vac}}(\mu), & \beta_L \beta_R > 1, \\ -2\pi^2 \left(\frac{1}{\beta_L} + \frac{1}{\beta_R} \right) RE_{\text{vac}} \left(\frac{4\pi^2}{\beta_L \beta_R} \mu \right), & \beta_L \beta_R < 1. \end{cases}$$

Datta & Jiang, [1806.07426](https://arxiv.org/abs/1806.07426) [22]

Apolo, Song & Yu, [2301.04153](https://arxiv.org/abs/2301.04153) [23]

cf. Hartman, Keller & Stoica, [1405.5137](https://arxiv.org/abs/1405.5137) [24]

- Sphere:** maximally symmetric,

Donnelly & Shyam, [1806.07444](https://arxiv.org/abs/1806.07444) [25]

$$\langle T_{ij} \rangle = -\frac{1}{4\pi\mu} \left(1 - \sqrt{1 - \frac{c\mu}{3R^2}} \right) \gamma_{ij}. \quad (11)$$

Then the trace relation $(-R) \partial_R \log Z_{T\bar{T}}(\mu) = \int d^2x \sqrt{\gamma} \langle T^i_i \rangle$ along with the flow equation (2) admits the general solution with a μ -independent integration constant a :

$$\log Z_{T\bar{T}}(\mu, a) = \frac{c}{3} \log \left[\frac{R}{a} \left(1 + \sqrt{1 - \frac{c\mu}{3R^2}} \right) \right] - \frac{R^2}{\mu} \sqrt{1 - \frac{c\mu}{3R^2}} + \frac{R^2}{\mu}$$

[25] is recovered with $a = \sqrt{c|\mu|/3}$ but one can also take $a = \epsilon$, thus decoupling the renormalization length scale ϵ with the deformation μ . In general the space of deformed theories can be parametrized by independent (μ, a) .

The log counterterm in (10) makes a crucial contribution to the on-shell action; it guarantees that the partition function is compatible with the $T\bar{T}$ differential equation, which is not the case for [25].

Caputa, Datta, Jiang & Kraus, 2011.04664 [26]
Li, 2012.14414 [27]

- Following these results, we would like to understand the entanglement structure of $T\bar{T}$ deformed theories

with the help of bulk geometry in future studies.

Lewkowycz, Liu, Silverstein & Torroba, 1909.13808 [28]