# Glue-on AdS holography for $T\bar{T}$-deformed CFTs

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## Foundations:
## Pure Einstein gravity in $\mathrm{AdS}_3$ / $\mathrm{CFT}_2$

*Fefferman & Graham*, 0710.0919 [1]
*Banados, Teitelboim & Zanelli*, hep-th/9204099 [2]
*Banados*, hep-th/9901148 [3]

We focus on pure Einstein gravity without matter in the bulk; the Einstein equations are solved by the metric:

$$\begin{aligned} ds^2 &= \ell^2\left(\frac{d\rho^2}{4\rho^2} + \frac{\big(du + \rho\,\bar{\mathcal{L}}(v)\,dv\big)\big(dv + \rho\,\mathcal{L}(u)\,du\big)}{\rho}\right) \\ &= n_\mu n_\nu dx^\mu dx^\nu + \frac{1}{\zeta}\gamma_{ij}dx^i dx^j, \quad \frac{1}{\zeta} \equiv \ell^{-2}g_{\varphi\varphi} \equiv r^2 \end{aligned} \tag{1}$$

The radial coordinate: $\rho \approx r^{-2} \equiv \zeta$, where $r$ is the "proper radius"; $\mathcal{L}(u)$ and $\bar{\mathcal{L}}(v)$ are arbitrary periodic functions.

The geometry is foliated by constant $\zeta$ surfaces $\mathcal{N}_\zeta$. The asymptotic boundary $\mathcal{N}_0$ is at $\zeta \to 0$, with coordinates: $x^i \sim \varphi, t$, while $u, v = \varphi \pm t$.

Our work is based on the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality:

*asympt.* $\mathrm{AdS}_3$ Gravity $\equiv$ Large $N$ $\mathrm{CFT}_2$ at $\mathcal{N}_0$

*Maldacena*, hep-th/9711200 [4]
*Aharony, Gubser, Maldacena, Ooguri & Oz*, hep-th/9905111 [5] *et al*

## $T\bar{T}$ deformations & cutoff $\mathrm{AdS}_3$

*Zamolodchikov*, hep-th/0401146 [6]
*Dubovsky, Flauger & Gorbenko*, 1205.6805 [7]
*Dubovsky, Gorbenko & Mirbabayi*, 1305.6939 [8]
*Smirnov & Zamolodchikov*, 1608.05499 [9]
*Cavaglià, Negro, Szécsényi & Tateo*, 1608.05534 [10]
*Dubovsky, Gorbenko & Mirbabayi*, 1706.06604 [11] *et al*

- Define "$T\bar{T}$" as the flow of action:

$$\partial_\mu I = 8\pi \int d^2x\, T\bar{T}_{(\mu)} = \pi \int d^2x\, \big(T^{ij}T_{ij} - (T^i_i)^2\big)_{(\mu)} \tag{2}$$

  where $x, \bar{x} = \varphi' \pm t'$. We shall start from $\mathrm{CFT}_2$ such that $T\bar{T}_{(\mu=0)} = T_{xx}T_{\bar{x}\bar{x}}$, but note that the deformation is *irrelevant*: conformal symmetry will be broken for $\mu \neq 0$.

- **Solvable:** the deformed spectrum of $\hat{H}(\mu)$ and $\hat{J}(\mu)$ on a cylinder of radius $R$ is a simple function:

$$\begin{aligned} E(\mu) &= -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu}{R}E(0) + \frac{4\mu^2}{R^4}J(0)^2}\right), \\ J(\mu) &= J(0) \end{aligned} \tag{3}$$

  of the undeformed spectrum $E(0)$, $J(0)$, under reasonable assumptions (e.g. translation invariance).

## Cutoff $\mathrm{AdS}_3$ duality:
## Holography within a finite Dirichlet wall

*McGough, Mezei & Verlinde*, 1611.03470 [12]
*Kraus, Liu & Marolf*, 1801.02714 [13] *et al*

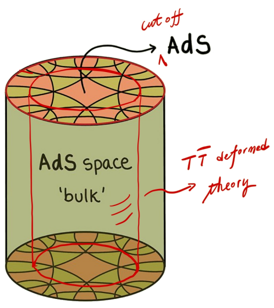



Image courtesy: *Aldegunde*, 2022 [14]

**Dictionary:** the $T\bar{T}$ deformed theory lives on $\mathcal{N}_{\zeta_c}$ with metric $\gamma_{ij}$. Location (radius) of the cutoff surface gets mapped to the deformation parameter $\mu$:

$$\zeta_c = -\frac{c\mu}{3\ell^2} \tag{4}$$

$T\bar{T}$ flow is recast geometrically as the $i, j$ components of the Einstein equations.

- **Significance:** gravity inside a finite box, dual to some "solvable" no-longer-conformal theory.

  *A step towards quantum gravity in reality$^{TM}$!*

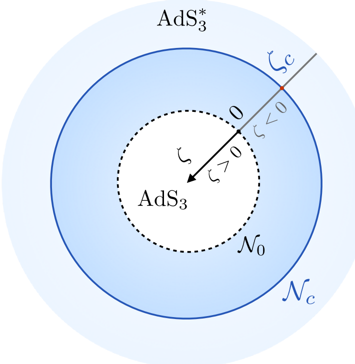
- **Caveat:** the duality only admits $\zeta_c > 0$ so $\mu < 0$. But $T\bar{T}$ itself admits $\mu > 0$ with nice properties.

  For comparison, the related proposal of *Guica & Monten*, 1906.11251 [15] admits both signs of $\mu$.

  *What is the other side of the duality?*

## Proposal:
## Glue-on $\mathrm{AdS}_3$ – beyond the infinity

Cutoff / *glue-on* $\mathrm{AdS}_{d+1}$ Gravity $\equiv$ $T\bar{T}$ deformed $\mathrm{CFT}_d$ at $\mathcal{N}_{\zeta_c}$




Top-down view of a constant $t$ slice

- **Analytic continuation:** metric (1) has only simple poles, thus admits well-defined analytic continuation:

$$\frac{1}{\zeta} \equiv \ell^{-2}g_{\varphi\varphi} \in \mathbb{R}, \qquad \zeta_c = -\frac{c\mu}{3\ell^2} \in \mathbb{R} \tag{5}$$

  What are we doing other than copy-pasting?

  - Physical quantities diverge as $\rho \to 0$. A prescription is required to "renormalize" the divergences.
  - Treatment for the singularity at $\zeta \to 0^\pm$: introduce $\mathcal{N}_{\zeta=\pm\epsilon}$ and glue them together ("**glue-on**"); exclude the $-\epsilon < \zeta < \epsilon$ region until finally sending $\epsilon \to 0$.
  - We formally denote the boundary surface by the limit $\mathcal{N}_0 = \mathcal{N}_{0^+} = \mathcal{N}_{0^-}$, though we need to keep track of the asymptotic cutoff $\epsilon$ in actual computations.

- **Extending the flow:** matching energy-momentum (Brown-York) & the $T\bar{T}$ flow equations:

$$\begin{aligned} T_{ij} &= \frac{\sigma_{\zeta_c}}{8\pi G}\left(K_{ij} - K\gamma_{ij} + \frac{1}{\ell|\zeta_c|}\gamma_{ij}\right), \\ \sigma_{\zeta_c} &= \frac{|\zeta_c|}{\zeta_c} = \pm 1, \\ \gamma_{ij} &= \zeta_c h_{ij}, \end{aligned} \tag{6}$$

  where $h_{ij}$ is the induced metric.

  The field theory metric $\gamma_{ij}$ is always positive-definite, while $h_{ij}$ becomes negative-definite for the glue-on region $\rho, \zeta < 0$. This discrepancy is the origin of the $|\zeta_c|$.

## $T\bar{T}$ physics from the glue-on geometry

- Demanding the extended geometry to be non-singular re-produces bounds on the $T\bar{T}$ deformed theory, e.g.

$$\zeta_c \geq -1 \quad \Leftrightarrow \quad \mu \leq \frac{3\ell^2}{c} \tag{7}$$

  This is a Killing horizon of the glue-on geometry; $(\det g_{\mu\nu})_{\zeta=-1} = 0$.

- **Spectrum from the extended geometry:** conserved charges of the Killing symmetries, can be computed with the *covariant formalism*:

  *Iyer & Wald*, gr-qc/9403028 [16]
  *Barnich & Brandt*, hep-th/0111246 [17]
  With $T\bar{T}$: *Kraus, Monten & Myers*, 2103.13398 [18]

$$\begin{aligned} \delta E(\mu) &\equiv \ell^{-1}\delta\mathcal{Q}_{\partial_{t'}}, \\ \delta J(\mu) &\equiv \delta\mathcal{Q}_{\partial_{\varphi'}}, \end{aligned}$$

  This reproduces $E(\mu)$, $J(\mu)$ with $\mu \in \mathbb{R}$ in (3).

- We are careful to match the bulk & boundary symmetries at $\mathcal{N}_{\zeta_c}$. One needs to choose the appropriate boundary coordinates $(t', \varphi')$, such that

$$ds_c^2 = \ell^2(-dt'^2 + d\varphi'^2), \qquad \varphi' \sim \varphi' + 2\pi. \tag{8}$$

  This is realized by:

$$\begin{aligned} dt' &= \sqrt{\big(1 - \zeta_c(T_u + T_v)^2\big)\big(1 - \zeta_c(T_u - T_v)^2\big)}\,dt, \\ d\varphi' &= d\varphi + \zeta_c\big(T_u^2 - T_v^2\big)\,dt. \end{aligned}$$

- The above ***state-dependent* map of coordinates** leads to the correct charges, the modified signal propagation speed $v'_\pm \equiv \pm\frac{d\varphi'}{dt'}$, and the $T\bar{T}$ **thermodynamics** when Wick rotated. In particular,

$$\mu > 0, \qquad T_L(\mu)\,T_R(\mu) \leq -\frac{1}{4\pi^2\ell^2\zeta_c} = \frac{3}{4\pi^2 c\mu} = T_H(\mu)^2. \tag{9}$$

  $T_{L,R}$ are temperatures associated with $u', v' = \varphi' \pm t'$.

  $T_H$ is the *Hagedorn temperature*: exceeding $T_H$ corresponds to a complex entropy.

  *Giveon, Itzhaki & Kutasov*, 1701.05576 [19]
  *Apolo, Detournay & Song*, 1911.12359 [20]

## $T\bar{T}$ partition functions from bulk actions

*Caputa, Datta, Jiang & Kraus*, 2011.04664 [21]

- The $T\bar{T}$ partition function is approximated by the bulk on-shell action of the dominant saddle:

$$Z_{T\bar{T}}(\mu) = \mathcal{Z}(\zeta_c) \approx e^{-I(\zeta_c)} \tag{10}$$

- $I(\zeta_c)$ is roughly the volume of the geometry, which diverges at $\zeta \to 0^\pm$. We apply a natural extension of *holographic renormalization* to remove the divergence.

- $I = -\log\mathcal{Z}$ satisfies the $T\bar{T}$ flow (2). This is non-trivial because we've packaged the quantum corrections of the boundary theory.

## Comparison with field theory analysis

- **Torus:** Large $c$, *modular invariance* & sparseness of the "light" spectrum:

  *Datta & Jiang*, 1806.07426 [22]
  *Apolo, Song & Yu*, 2301.04153 [23]
  cf. *Hartman, Keller & Stoica*, 1405.5137 [24]

$$\log Z_{T\bar{T}}(\mu) \approx \begin{cases} -\dfrac{1}{2}\,(\beta_L + \beta_R)\,RE_{\mathrm{vac}}(\mu), & \beta_L\beta_R > 1 \\ -2\pi^2\left(\dfrac{1}{\beta_L} + \dfrac{1}{\beta_R}\right)RE_{\mathrm{vac}}\left(\dfrac{4\pi^2}{\beta_L\beta_R}\mu\right), & \beta_L\beta_R < 1 \end{cases}$$

- **Sphere:** maximally symmetric, *Donnelly & Shyam*, 1806.07444 [2

$$\langle T_{ij}\rangle = -\frac{1}{4\pi\mu}\left(1 - \sqrt{1 - \frac{c\mu}{3L^2}}\right)\gamma_{ij}. \tag{11}$$

  *Donnelly & Shyam*, 1806.07444 [25] and *Li*, 2012.14414 [26]

- Given the explicit stress tensor $\langle T_{ij}\rangle \propto \gamma_{ij}$, the flow equations:

$$\begin{aligned} \partial_\mu \log Z_{T\bar{T}}(\mu) &= 8\pi \int d^2x\sqrt{\gamma}\,\langle T\bar{T}\rangle \\ -L\,\partial_L \log Z_{T\bar{T}}(\mu) &= \int d^2x\sqrt{\gamma}\,\langle T^i_i\rangle \end{aligned}$$

- ... admit the general solution with a *$\mu$-independent* integration $a$:

$$\log Z_{T\bar{T}}(\mu, a) = \frac{c}{3}\log\left[\frac{L}{a}\left(1+\sqrt{1-\frac{c\mu}{3L^2}}\right)\right] - \frac{L^2}{\mu}\sqrt{1-\frac{c\mu}{3L^2}} + \frac{L^2}{\mu}$$

- *Donnelly & Shyam*, 1806.07444 [25] is recovered with $a = \sqrt{c|\mu|/3}$ but one can also take $a = \epsilon$, thus decoupling the energy (RG) scale $\epsilon$ with the deformation $\mu$.

## Gravitational actions: with counterterms

*Caputa, Datta, Jiang & Kraus*, 2011.04664 [21], *Donnelly & Shyam*, 1806.07444 [25], and *Li*, 2012.14414 [26]

- Gravitational actions, with counterterms:

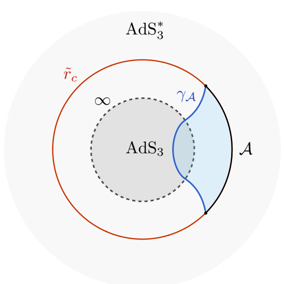
$$I_{\mathcal{M}}(\zeta_1, \zeta_2) := -\frac{1}{16\pi G}\int_{\zeta_1}^{\zeta_2} d\zeta \int d^2x\sqrt{g}\,(R + 2\ell^{-2}),$$

$$I_{\mathcal{N}_\zeta} := -\frac{1}{8\pi G}\int_{\mathcal{N}_\zeta} d^2x\sqrt{h}\,h^{ij}K_{ij} + \frac{\sigma_\zeta}{8\pi G}\int_{\mathcal{N}_\zeta} d^2x\sqrt{h}\left(\ell^{-1} - \frac{\ell\mathcal{R}[h]}{4}\log|\zeta|\right),$$

  ... consistent with the Brown-York stress tensor $T_{ij}$, fully renormalized at $\zeta \to 0^\pm$.

- The lack of diff invariance of the $\log|\zeta|$ counterterm is known to be a reflection of the Weyl anomaly: *Henningson & Skenderis*, *de Haro, Solodukhin & Skenderis*, *Papadimitriou*.
- The log counterterm makes a crucial contribution to the on-shell action; it guarantees that the partition function is compatible with the $T\bar{T}$ differential equation — not the case for *Donnelly & Shyam*.
- In general the space of deformed theories are parametrized by $(\mu, a)$, where $a$ is the length scale (inverse energy scale). In many contexts they are naturally related: $a = \sqrt{c|\mu|/3}$, but $a$ can be tuned by RG.

## Summary & future directions




RT surface for the extended $\mathrm{AdS}_3$

- Holographic proposal for $T\bar{T}$-deformed CFTs with $\mu \in \mathbb{R}$.
- As evidence, we show that the $T\bar{T}$ trace flow equation, the spectrum on the cylinder, and the partition function on the torus and the sphere, among other results, can all be reproduced from bulk calculations in glue-on $\mathrm{AdS}_3$.
- We hope to understand the entanglement structure of $T\bar{T}$ deformed theories from bulk geometry.
  - *Donnelly & Shyam*, 1806.07444 [25] and *Lewkowycz, Liu, Silverstein & Torroba*, 1909.13808 [30]
  - *He & Sun*, 2301.04435 [31], *He, Yang, Zhang & Zhao*, 2305.10984 [32], *Tian*, 2306.01258 [33], and *Hou, He & Jiang*, 2306.07784 [34]

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