

Thermal propagator for harmonic oscillators

$$G(\tau) = \frac{1}{Z} \text{Tr} (e^{-\beta H} x(\tau) x(0)) \quad (1)$$

The Hamiltonian is $H = \omega(a^\dagger a + \frac{1}{2})$, while $x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger)$.

Operator formalism

Mode expansion of Lorentzian operator in the Heisenberg picture:

$$x(t) = e^{iHt} x(0) e^{-iHt} = \frac{1}{\sqrt{2m\omega}} (a e^{-i\omega t} + a^\dagger e^{i\omega t}) \quad (2)$$

This follows from the commutation relation between $a^{(\dagger)}$ and H , or from the quantum EOM in the Heisenberg picture. Wick-rotation to Euclidean signature: $\tau = it$, $t = -i\tau$, and we have:

$$x(\tau) = e^{H\tau} x(0) e^{-H\tau} = \frac{1}{\sqrt{2m\omega}} (a e^{-\omega\tau} + a^\dagger e^{\omega\tau}) \quad (3)$$

Insert $x(\tau) x(0)$ into the definition of $G(\tau)$, and note that only aa^\dagger and $a^\dagger a$ contribute to non-zero amplitudes, then we have:

$$\begin{aligned} G(\tau) &= \frac{1}{Z} \sum_{n=0}^{\infty} \left\langle n \left| e^{-\beta\omega(n+\frac{1}{2})} \frac{aa^\dagger e^{-\omega\tau} + a^\dagger a e^{\omega\tau}}{2m\omega} \right| n \right\rangle \\ &= \frac{1}{Z} \sum_{n=0}^{\infty} \left\langle n \left| e^{-\beta\omega(n+\frac{1}{2})} \frac{(n+1)e^{-\omega\tau} + n e^{\omega\tau}}{2m\omega} \right| n \right\rangle \end{aligned} \quad (4)$$

Note that:

$$\sum_{n=0}^{\infty} n e^{-\beta\omega n} = -\frac{1}{\beta} \frac{\partial}{\partial \omega} \sum_{n=0}^{\infty} e^{-\beta\omega n} \quad (5)$$

The summations in $G(\tau)$ can be completed, and we get:

$$G(\tau) = \frac{1}{2m\omega} \frac{\cos((\frac{\beta}{2} - \tau)\omega)}{\sin(\frac{\beta\omega}{2})} \quad (6)$$

Path Integral

The mathematical trick in (5) is crucial in our path integral derivation of $G(\tau)$. We have:

$$G(\tau) = \langle x(\tau) x(0) \rangle = \frac{1}{Z} \int \mathcal{D}x e^{-S} x(\tau) x(0), \quad (7)$$

$$S = \int_0^\beta dt \mathcal{L}_E, \quad \mathcal{L}_E = \frac{1}{2} m \dot{x}(\tau)^2 + \frac{1}{2} m \omega^2 x(\tau)^2 \quad (8)$$

Note that the path integral of a total derivative vanishes: $0 = \int \mathcal{D}x \frac{\delta}{\delta x}$ (see *Polchinski*), we have:

$$0 = \int \mathcal{D}x \frac{\delta}{\delta x(\tau)} \left\{ e^{-S} x(\tau') \right\} = \int \mathcal{D}x e^{-S} \left\{ -\frac{\delta S}{\delta x(\tau)} x(\tau') + \delta(\tau - \tau') \right\} \quad (9)$$

$\frac{\delta S}{\delta x(\tau)}$ yields precisely the quantum EOM:

$$\frac{\delta S}{\delta x(\tau)} = m \left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) x(\tau) \quad (10)$$

After some re-organization, we get the quantum equation for the Green's function (which, for a free theory, is identical to the classical one):

$$m \left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) \int \mathcal{D}x e^{-S} x(\tau) x(\tau') = \delta(\tau - \tau') \int \mathcal{D}x e^{-S}, \quad (11)$$

$$\left(-\frac{\partial^2}{\partial \tau^2} + \omega^2 \right) G(\tau, \tau') = \frac{1}{m} \delta(\tau - \tau'), \quad G(\tau) = G(\tau, 0) \quad (12)$$

Note that $\tau \in [0, \beta]$ and $x(\tau)$ satisfies the periodic boundary condition: $x(\beta) = x(0)$, hence also $G(\beta) = G(0)$; we can extend the domain so that $x(\tau)$ is a function with period β , defined for $\tau \in \mathbb{R}$. Furthermore, note that by definition $G(\tau, \tau') = G(\tau', \tau)$; this is the Feynman propagator! Integrating around $\tau = \tau'$ then setting $\tau' = 0$ gives:

$$G'(\beta^-) - G'(0^+) = \frac{1}{m} \quad (13)$$

While $G(\tau)$ away from $\tau = 0$ grows (or decays) exponentially: $G(\tau) \propto e^{\omega\tau} + e^{\omega(\beta-\tau)}$. This fixes $G(\tau)$ uniquely:

$$G(\tau) = \frac{1}{2m\omega} \frac{\cos\left(\left(\frac{\beta}{2} - \tau\right)\omega\right)}{\sin\left(\frac{\beta\omega}{2}\right)} \quad (14)$$