

1 Spacelike Geodesic for Poincaré AdS

The Poincaré AdS_{d+1} metric is given by:

$$ds^2 = G_{IJ} dX^I dX^J = \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2} = \frac{dx^2 + dz^2}{z^2}, \quad (1.1)$$

$$dx^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad X^I \sim (t, \vec{x}, z) \sim (x^\mu, z), \quad (1.2)$$

Note that here we define dx^2 using the flat metric η , while dX has an index that should be raised and lowered using the curved metric $G_{IJ} = \frac{1}{z^2} \eta_{IJ}$. Generally upper case tensors are handled with G_{IJ} , while lower case ones are handled with η . The full isometry of such spacetime is then given by $SO(d, 2)$ with $\frac{(d+2)(d+1)}{2}$ generators, including:

1. Among the $x^\mu \sim (x^0, \vec{x}) \equiv (t, \vec{x})$ directions:
 - (a) $\frac{d(d-1)}{2}$ $SO(d-1, 1)$ rotations $x_\mu \partial_\nu - x_\nu \partial_\mu$, boosts included;
 - (b) d translations ∂_μ ;
2. Dilation with $X^I \sim (x^\mu, z) \mapsto \lambda X^I = \lambda(x^\mu, z)$, generated by $\Delta = X^I \partial_I = z \partial_z + x^\mu \partial_\mu$;
3. Special conformal transformations; see [Appendix A](#) for an intuitive derivation. We have:

$$k_\mu = (z^2 + x^2) \partial_\mu - 2x_\mu \Delta \quad (1.3)$$

By Noether's theorem, $Q_\Xi = V_I \Xi^I = G_{IJ} \Xi^I V^J$ is conserved along the geodesic; here V^I is the normalized tangent vector, while Ξ^I is some Killing vector of the spacetime, e.g. one from the list above. We can then write down the conserved charges along a geodesic γ in Poincaré AdS:

$$\begin{aligned} p_\mu &= G_{\mu I} V^I = V_\mu \\ m_{\mu\nu} &= x_\mu V_\nu - x_\nu V_\mu \\ \Delta &= X^I V_I = \frac{z V^z}{z^2} + x^\mu p_\mu \\ k_\mu &= (z^2 + x^2) p_\mu - 2x_\mu \Delta \end{aligned} \quad (1.4)$$

These are all integration constants along γ . Again note that X, V have indices that should be handled with G_{IJ} , in particular,

$$dX_\mu = G_{\mu I} dX^I = \frac{1}{z^2} dx_\mu, \quad V^\mu = \frac{dX^\mu}{d\lambda} = z^2 p^\mu \quad (1.5)$$

On the other hand, V^μ should be properly normalized, therefore:

$$\|V\|^2 = g_{IJ} V^I V^J = \frac{(V^z)^2}{z^2} + z^2 p^2 = \delta = 0, \pm 1, \quad (1.6)$$

For $p^2 = 0$, we have $\|V\|^2 = \frac{(V^z)^2}{z^2} \geq 0$, thus γ can be either spacelike or null, but not timelike. In fact, we have $\frac{V^z}{z} = \frac{1}{z} \frac{dz}{d\lambda} = 0, \pm 1$; along with $\frac{dx^\mu}{d\lambda} = z^2 p^\mu$, we obtain:

$$\begin{aligned} p^2 = 0, \quad \text{spacelike: } & z(\lambda) = z(0) e^{\pm\lambda}, \quad x^\mu(\lambda) = x^\mu(0) \pm z(0)^2 p^\mu \frac{e^{\pm 2\lambda} - 1}{2}, \\ & \text{null: } z(\lambda) = z(0), \quad x^\mu(\lambda) = x^\mu(0) + z(0)^2 p^\mu \lambda, \end{aligned} \quad (1.7)$$

From now on we shall focus on the $p^2 \neq 0$ situation. Note that we can complete the square on the right-hand side of the k^μ conservation such that:

$$k_\mu + p_\mu a^2 = p_\mu(z^2 + (x - a)^2), \quad \Delta = p_\mu a^\mu \quad (1.8)$$

For $p^\mu \neq 0$, we can always find some c_μ such that $c_\mu p^\mu \neq 0$, therefore we have:

$$z^2 + (x - a)^2 = \frac{c_\mu k^\mu}{c_\mu p^\mu} + a^2 = \left(\frac{L}{2}\right)^2 = \text{const.}, \quad \Delta = p_\mu a^\mu \quad (1.9)$$

We see that γ lands on a “sphere” centered at $z = 0, x = a$ in \mathbb{R}^{d+1} ; with the Lorentzian metric, it is actually a hyperboloid. Note that for now the radius $\left(\frac{L}{2}\right)^2$ can actually be negative or zero; more specifically,

1. $L^2 > 0$: one-sheet hyperboloid;
2. $L^2 < 0$: two-sheet hyperboloid;
3. $L^2 = 0$: conic surface.

On the other hand, we can actually solve X^I completely by combining (1.5), (1.6); we have:

$$\frac{dx^\mu}{d\lambda} = z^2 p^\mu, \quad \frac{dz}{d\lambda} = \pm z \sqrt{\delta - z^2 p^2}, \quad \frac{dx^\mu}{dz} = \pm \frac{z p^\mu}{\sqrt{\delta - z^2 p^2}}, \quad \delta - z^2 p^2 = \left(\frac{V^z}{z}\right)^2 \geq 0 \quad (1.10)$$

$$\implies \gamma \subset x^\mu = a^\mu \pm \frac{p^\mu}{p^2} \sqrt{\delta - z^2 p^2}, \quad z^2 + (x - a)^2 = \frac{\delta}{p^2} = \left(\frac{L}{2}\right)^2 \quad (1.11)$$

This confirms our observation above, and further reveals that γ lies in a “plane” in the \mathbb{R}^d subspace consisting of the x^μ coordinates. The behavior of γ is sensitive to the sign of δ and p^2 ; more specifically,

1. $\delta = -1$, i.e. for timelike γ , we must have $p^2 < 0$, and $z \geq \frac{1}{\|p\|} = \frac{L}{2} > 0$, namely z is bounded from below. This means that timelike geodesic can never reach the asymptotic boundary $z \rightarrow 0$. In this case, γ is a section of the one-sheet hyperboloid.
2. $\delta = +1$, i.e. for spacelike γ , we can have $p^2 > 0$ or $p^2 < 0$.
 - (a) For $p^2 > 0$, again we have $\frac{L}{2} = \frac{1}{\|p\|} > 0$, and γ is again a cross-section of the one-sheet hyperboloid. However, now we have $z \leq \frac{L}{2}$, namely z is bounded from above, and:

$$z \rightarrow 0, \quad x^\mu = a^\mu \pm \frac{p^\mu}{p^2} = a^\mu \pm \frac{L}{2} \hat{p}^\mu, \quad \hat{p}^\mu = \frac{p^\mu}{\|p\|} \quad (1.12)$$

We can also nicely parametrize X^I in terms of the proper length λ ; for convenience, set $x^\mu(0) = a^\mu, z(0) = \frac{L}{2}$, then we have:

$$z(\lambda) = \frac{L}{2} \frac{1}{\cosh \lambda}, \quad x^\mu(\lambda) = a^\mu \pm \hat{p}^\mu \frac{L}{2} \tanh \lambda \quad (1.13)$$

- (b) For $p^2 < 0$, we have $\left(\frac{L}{2}\right)^2 = \frac{1}{p^2} < 0$, and γ is now a cross-section of the two-sheet hyperboloid. Again as $z \rightarrow 0$ it lands at $x^\mu = a^\mu \pm \frac{L}{2} \hat{p}^\mu$; however, $|x^\mu|$ grows with z and extends into the bulk instead of returning to the boundary, i.e. $z, |x^\mu| \rightarrow \infty$.

The differential equation in this case is almost the same as in (a), but now we have to choose a different initial condition, since γ won't even reach $x = a$. However, we can

actually set $x(0), z(0) \rightarrow \infty$; we then have:

$$z(\lambda) = \left| \frac{L}{2} \right| \frac{1}{\sinh \lambda}, \quad x^\mu(\lambda) = a^\mu \pm \widehat{p}^\mu \left| \frac{L}{2} \right| \coth \lambda \quad (1.14)$$

This is an evidence that $z \rightarrow \infty$ might not be the “end of the world” after all; it’s likely that $z \rightarrow \infty$ is only a horizon, since a spacelike geodesic γ can reach it within finite proper length λ .

In summary, there are 3 types of spacelike geodesics in Poincaré AdS, which closely resemble the 3 types of conic sections:

1. $p^2 = 0$: parabolic, given in (1.7), with one end going to $z \rightarrow 0$ and the other end going to $z \rightarrow \infty$; it takes infinite proper length λ for it to reach 0 or ∞ .
2. $p^2 < 0$: hyperbolic, given in (1.11) and (1.14), also with one end going to $z \rightarrow 0$ and the other end going to $z \rightarrow \infty$, but it reaches ∞ within finite λ .
3. $p^2 > 0$: elliptic, given in (1.11) and (1.13), with both ends going to $z \rightarrow 0$, and it takes infinite proper length λ for it to reach either end.

Now consider two points x_1, x_2 near the boundary $z = \epsilon$ connected by a spacelike geodesic γ . This can only be the “elliptic type” discussed above. We have:

$$x^\mu = a^\mu \pm \widehat{p}^\mu \sqrt{\left(\frac{L}{2}\right)^2 - z^2}, \quad z^2 + (x - a)^2 = \left(\frac{L}{2}\right)^2, \quad (1.15)$$

$$a = \frac{x_1 + x_2}{2}, \quad \widehat{p} \propto x_2 - x_1 \quad (1.16)$$

It’s length is then given by:

$$\begin{aligned} A &= \int_{z \geq \epsilon} \sqrt{\frac{dx^2 + dz^2}{z^2}} \\ &= \int_{|\lambda| \leq \Lambda} \sqrt{\frac{(d \tanh \lambda)^2 + (d \operatorname{sech} \lambda)^2}{(\operatorname{sech} \lambda)^2}}, \quad \Lambda = \cosh^{-1}\left(\frac{L}{2\epsilon}\right) \\ &= \int_{-\Lambda}^{\Lambda} d\lambda = 2\Lambda = 2 \cosh^{-1}\left(\frac{L}{2\epsilon}\right) \sim 2 \log \frac{L}{\epsilon}, \quad \epsilon \rightarrow 0 \end{aligned} \quad (1.17)$$

2 Einbein Action

The einbein action of a point particle is given by:

$$S[\eta, X] = \frac{1}{2} \int d\tau \left(\eta^{-1} \dot{X}_\mu \dot{X}^\mu - \eta m^2 \right) \quad (2.1)$$

Under worldline reparametrization: $\tau \mapsto \tau' = f(\tau)$, we have $X'(\tau') = X(\tau)$, i.e. X^μ transforms like a **scalar** under worldline diffeomorphism; $X(\tau) \mapsto X'(\tau) = X(f^{-1}(\tau))$.

On the other hand, η should be treated like an einbein: $\eta = \sqrt{-\gamma}$, here $\gamma = \gamma_{\tau\tau}$ is the worldline metric; $\gamma < 0$ due to the Lorentzian signature [1]. We have:

$$\eta = \sqrt{-\gamma} \mapsto \eta' = \eta \det \frac{\partial \tau}{\partial \tau'} = \eta \frac{\partial \tau}{\partial \tau'}, \quad (2.2)$$

$$\eta^{-1} = -\sqrt{-\gamma} \gamma^{-1} \mapsto (\eta')^{-1} = \eta^{-1} \left(\frac{\partial \tau}{\partial \tau'} \right) \left(\frac{\partial \tau'}{\partial \tau} \frac{\partial \tau'}{\partial \tau} \right) = \eta^{-1} \frac{\partial \tau'}{\partial \tau} \quad (2.3)$$

It is clear that the action is invariant under the transformation:

$$\begin{aligned} S' &= \frac{1}{2} \int d\tau' \left((\eta')^{-1} \partial_{\tau'} X_\mu \partial_{\tau'} X^\mu - (\eta') m^2 \right) \\ &= \frac{1}{2} \int d\tau \frac{\partial \tau'}{\partial \tau} \left(\eta^{-1} \frac{\partial \tau'}{\partial \tau} \cdot \frac{\partial \tau}{\partial \tau'} \partial_\tau X_\mu \cdot \frac{\partial \tau}{\partial \tau'} \partial_\tau X^\mu - \eta \frac{\partial \tau}{\partial \tau'} m^2 \right) = S \end{aligned} \quad (2.4)$$

We can eliminate η classically by placing it on shell:

$$0 = \frac{\delta S}{\delta \eta} = -\eta^{-2} \dot{X}_\mu \dot{X}^\mu - m^2, \quad \eta[X] = \frac{1}{m} \sqrt{-\dot{X}_\mu \dot{X}^\mu} \quad (2.5)$$

Substitute this back to the action, and we have:

$$\begin{aligned} S[X] &= S[\eta = \eta[X], X] = \frac{1}{2} \int d\tau \left(m (-\dot{X}_\mu \dot{X}^\mu)^{-\frac{1}{2}} \dot{X}_\mu \dot{X}^\mu - m (-\dot{X}_\mu \dot{X}^\mu)^{+\frac{1}{2}} \right) \\ &= -m \int d\tau \sqrt{-\dot{X}_\mu \dot{X}^\mu} \end{aligned} \quad (2.6)$$

3 Ricci Tensor for Static Spherical Metric

Consider the metric:

$$ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = \eta_{ab} e^a e^b \quad (3.1)$$

Here we've defined the following vierbein:

$$\begin{aligned} e^t &= \sqrt{f(r)} dt \\ e^r &= \sqrt{h(r)} dr \\ e^\theta &= r d\theta \\ e^\phi &= r \sin \theta d\phi \end{aligned} \quad (3.2)$$

To find the connection form, we first note the *Cartan's structure equation* [2]:

$$\begin{aligned} de^a &= d(e_\nu^a dx^\nu) = \partial_\mu e_\nu^a dx^\mu \wedge dx^\nu \\ &= (-\omega_{\mu b}^a e_\nu^b + \Gamma_{\mu\nu}^\lambda e_\lambda^a) dx^\mu \wedge dx^\nu \\ &= -\omega_{\mu b}^a \wedge e^b + T^a \end{aligned} \quad (3.3)$$

Where T^a is the torsion tensor. Here we've used the relation between spin connection $\omega_{\mu b}^a$ and the affine connection $\Gamma_{\mu\nu}^\lambda$. For the torsion-free Levi-Civita connection, we have $de^a = -\omega_{\mu b}^a \wedge e^b$.

Rigidity of the veilbein, namely $\nabla_\mu (g_{\mu\nu} e_a^\mu e_b^\nu) = \nabla_\mu \eta_{ab} = 0$, further implies that:

$$\omega_{\mu ab} = -\omega_{\mu ba} \quad (3.4)$$

With these constraints we can solve for the connection form, and we find that only the following components are non-vanishing¹:

$$\omega_r^t = \frac{f'}{2\sqrt{fh}} dt = -\omega_t^r \quad (3.5)$$

¹ Go to <https://github.com/bryango/Archive/blob/master/HW-Gravity/gravity1/nb/vielbein.wl> for a Mathematica script for this calculation.

$$\omega^r_\theta = \frac{-1}{\sqrt{h}} d\theta = -\omega^\theta_r \quad (3.6)$$

$$\omega^r_\phi = \frac{-\sin\theta}{\sqrt{h}} d\phi = -\omega^\phi_r \quad (3.7)$$

$$\omega^\theta_\phi = -\cos\theta d\phi = -\omega^\phi_\theta \quad (3.8)$$

The curvature form is thus given by:

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \quad (3.9)$$

The exterior derivative is easily computed in the dx^μ basis; however, to get to the Ricci tensor, it would be convenient to switch to the e^a basis. Here we only write down the upper half of Ω^a_b since the lower half can be inferred from anti-symmetry:

$$\Omega^a_b \sim \begin{pmatrix} 0 & R^t_{rtr} e^t \wedge e^r & R^t_{\theta t\theta} e^t \wedge e^\theta & R^t_{\phi t\phi} e^t \wedge e^\phi \\ \cdots & 0 & R^r_{\theta r\theta} e^r \wedge e^\theta & R^r_{\phi r\phi} e^r \wedge e^\phi \\ \cdots & \cdots & 0 & R^\theta_{\phi\theta\phi} e^\theta \wedge e^\phi \\ \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad (3.10)$$

$$\begin{aligned} R^t_{rtr} &= \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh}, \\ R^t_{\theta t\theta} &= \frac{-f'}{2rfh} = R^t_{\phi t\phi}, \\ R^r_{\theta r\theta} &= \frac{h'}{2rh^2} = R^r_{\phi r\phi}, \\ R^\theta_{\phi\theta\phi} &= \frac{h-1}{r^2h} \end{aligned} \quad (3.11)$$

The Ricci tensor in e^a basis is given by $R_{ab} = R^c_{acb}$. Note that the components are non-zero iff. $a = b$, i.e. the Ricci tensor is diagonal. We have:

$$\begin{aligned} R_{ab} &\sim \text{diag} (R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t}, \cdots, \cdots, \cdots) \\ &= \text{diag} (R^t_{rtr} + R^t_{\theta t\theta} + R^t_{\phi t\phi}, \cdots, \cdots, \cdots) \\ &= \text{diag} \left(R^t_{rtr} + R^t_{\theta t\theta} + R^t_{\phi t\phi}, \right. \\ &\quad \left. R^t_{rtr} + R^r_{\theta r\theta} + R^r_{\phi r\phi}, \right. \\ &\quad \left. R^t_{\theta t\theta} + R^r_{\theta r\theta} + R^\theta_{\phi\theta\phi}, \right. \\ &\quad \left. R^t_{\phi t\phi} + R^r_{\phi r\phi} + R^\theta_{\phi\theta\phi} \right) \\ &= \text{diag} \left(R^t_{rtr} + 2R^t_{\theta t\theta}, \right. \\ &\quad \left. R^t_{rtr} + 2R^r_{\theta r\theta}, \right. \\ &\quad \left. R_{\theta\theta} = R^t_{\theta t\theta} + R^r_{\theta r\theta} + R^\theta_{\phi\theta\phi}, \right. \\ &\quad \left. R_{\phi\phi} = R_{\theta\theta} \right) \\ &= \text{diag} \left(\frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh} - \frac{f'}{rfh}, \right. \\ &\quad \frac{f'^2}{4f^2h} + \frac{f'h'}{4fh^2} - \frac{f''}{2fh} + \frac{h'}{rh^2}, \\ &\quad R_{\theta\theta} = -\frac{f'}{2rfh} + \frac{h'}{2rh^2} + \frac{h-1}{r^2h}, \\ &\quad \left. R_{\phi\phi} = R_{\theta\theta} \right), \end{aligned} \quad (3.12)$$

To go back to the dx^μ basis, we have $R_{\mu\nu} = R_{ab}e_\mu^a e_\nu^b$. In particular, here we have $R_{\mu\mu} = R_{aa}(e_\mu^a)^2$, therefore:

$$R_{\mu\nu} = \text{diag} \left(\begin{aligned} &\frac{f'^2}{4fh} + \frac{f'h'}{4h^2} - \frac{f''}{2h} - \frac{f'}{rh}, \\ &\frac{f'^2}{4f^2} + \frac{f'h'}{4fh} - \frac{f''}{2f} + \frac{h'}{rh}, \\ &R_{\theta\theta} = -\frac{rf'}{2fh} + \frac{rh'}{2h^2} + \frac{h-1}{h}, \\ &R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta \end{aligned} \right) \quad (3.13)$$

4 Jackiw–Teitelboim Gravity

The Jackiw–Teitelboim (JT) action is given by:

$$S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi (R + 2) \quad (4.1)$$

$\frac{\delta S}{\delta \Phi} = 0$ gives us $R = -2$. Now consider $\frac{\delta S}{\delta g^{\mu\nu}}$, and we have:

$$\begin{aligned} \delta_g S &= \frac{1}{16\pi G} \int d^2x \Phi \delta(\sqrt{-g} (R + 2)), \quad R = g^{\mu\nu} R_{\mu\nu}, \\ &= \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi \left\{ \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + 2) \right) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right\} \end{aligned} \quad (4.2)$$

Note that the $g^{\mu\nu} \delta R_{\mu\nu}$ term is a total derivative²:

$$\begin{aligned} g^{\mu\nu} \delta R_{\mu\nu} &= (\nabla^\mu \nabla^\nu - g^{\mu\nu} \nabla^\lambda \nabla_\lambda) \delta g_{\mu\nu} \\ &= -(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda) \delta g^{\mu\nu} \\ &= \nabla_\lambda (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda - g^{\mu\lambda} \delta \Gamma_{\nu\mu}^\nu) \end{aligned} \quad (4.3)$$

In Einstein gravity this gets reduced to a boundary term. But here we have an additional factor of Φ , so after integration by parts, we actually get the equation of motion (EoM) for Φ , up to some boundary terms³:

$$\delta_g S \sim \frac{1}{16\pi G} \int d^2x \sqrt{-g} \left\{ \Phi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + 2) \right) - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda) \Phi \right\} \delta g^{\mu\nu} \quad (4.4)$$

In 2D, we have $R_{\mu\nu} \equiv \frac{1}{2} g_{\mu\nu} R$, so the EoM is simply:

$$(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^\lambda \nabla_\lambda + g_{\mu\nu}) \Phi = 0 \quad (4.5)$$

Contraction with $g^{\mu\nu}$ further gives us $(\nabla^\lambda \nabla_\lambda - 2) \Phi = 0$, so in the end we have:

$$(\nabla_\mu \nabla_\nu - g_{\mu\nu}) \Phi = 0, \quad R = -2 \quad (4.6)$$

² See the amazing lecture note by Matthias Blau at <http://www.blau.itp.unibe.ch/GRlecturenotes.html>.

³ See e.g. Section 2 of [3].

A Derivation of the special conformal transformations

Special conformal transformations can be understood as translations conjugated by *inversions*. Note that $\frac{dz^2}{z^2}$ is invariant under $z \mapsto \frac{1}{z}$; if we include the x^μ directions, we can consider:

$$\mathcal{I}: \chi^I \mapsto \frac{\chi^I}{\chi^2}, \quad \chi^2 = -t^2 + \vec{x}^2 + z^2, \quad (\text{A.1})$$

$$\mathcal{I}^2 = \mathbb{1}, \quad ds^2 \mapsto \left(\frac{\delta_J^I - 2 \frac{\chi^I \chi_J}{\chi^2}}{\chi^2} d\chi^J \right)^2 / \left(\frac{z}{\chi^2} \right)^2 = \frac{d\chi^2}{z^2} = ds^2 \quad (\text{A.2})$$

We see that inversion \mathcal{I} is indeed a (discrete) symmetry of the metric. Here we've defined yet another lower case variable $\chi^I \sim (x^\mu, z)$, which as a contravariant vector has the same components as X^I , but with an index that should be lowered by the flat metric η_{IJ} , i.e. $\chi_I = \eta_{IJ} \chi^J = \eta_{IJ} X^J$. The d special conformal generators are then given by:

$$\begin{aligned} k_\mu &= \frac{\partial}{\partial a^\mu} \left(\mathcal{I} \circ e^{a^\nu P_\nu} \circ \mathcal{I} \circ X^I \right)_{a=0} \frac{\partial}{\partial X^I} \\ &= \frac{\partial}{\partial a^\mu} \left(\frac{\frac{\chi^I}{\chi^2} + a^I}{\left| \frac{\chi^J}{\chi^2} + a^J \right|^2} \right)_{a=0} \frac{\partial}{\partial X^I} \\ &= \frac{\partial}{\partial a^\mu} \left(\frac{\chi^I + a^I \chi^2}{1 + 2a^I \chi_I + a^2 \chi^2} \right)_{a=0} \frac{\partial}{\partial X^I} \\ &= \chi^2 \partial_\mu - 2x_\mu X^I \partial_I \\ &= \chi^2 \partial_\mu - 2x_\mu \Delta \end{aligned} \quad (\text{A.3})$$

References

- [1] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, **December 1998**. ISBN: 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521-63303-1.
- [2] Sean M. Carroll. *Lecture notes on general relativity*, **December 1997**. arXiv: [gr-qc/9712019](https://arxiv.org/abs/gr-qc/9712019).
- [3] Daniel Harlow & Daniel Jafferis. *The Factorization Problem in Jackiw-Teitelboim Gravity*. *JHEP*. **02:177, 2020**. arXiv: [1804.01081](https://arxiv.org/abs/1804.01081) [[hep-th](https://arxiv.org/archive/hep)].