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1 Example of limit in <u>Vect</u>:

$$I = \left\{ \begin{array}{c} \bullet \\ \star \end{array} \right\} \xrightarrow{F} \left\{ \begin{array}{c} 0 \\ \mathbb{R} \end{array} \right\}$$

$$\supset \underline{\mathbf{Vect}} \colon \mathbb{R} \text{ vector space }, \tag{1}$$

(a) By definition, we have:

$$\begin{array}{ccc}
\Delta(\lim F) \\
& \exists ! \, \Delta(g) \\
& \downarrow \sigma \\
\Delta(V) & \stackrel{\eta}{=} & F
\end{array} \tag{2}$$

Where $g: V \to \lim F$. More intuitively, for the above $F: I \to \underline{\mathbf{Vect}}$, this translates to the following diagram in \mathbf{Vect} :

$$V \xrightarrow{\exists ! g} \lim F \downarrow 0$$

$$\downarrow 0$$

$$\downarrow$$

Now we verify that $\lim F = \mathbb{R}$, along with the following choice of σ :

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = \mathbb{1}_{\mathbb{R}} \tag{4}$$

In fact, for the above diagram (3) to be commutative, we must have $g = \eta_{\star}$. Note that such g is unique once σ is chosen; for our choice of σ , if $g \neq \eta_{\star}$, then the diagram *cannot* commute. Hence $\lim F = \mathbb{R}$, along with the above choice of $\sigma \colon \Delta(\mathbb{R}) \Rightarrow F$. In other words, we have:

$$\begin{cases}
\bullet, \star \\
 & \\
\Delta(\mathbb{R}) & \xrightarrow{\sigma} & F
\end{cases}$$

$$\mathbb{R} \qquad \qquad \begin{cases}
0, \mathbb{R} \end{cases}$$

(b)(c) From diagram (2) and discussions in (a), we know that:

$$\exists ! \ \tau = \Delta(g) \colon \ \Delta(V) \Longrightarrow \Delta(\mathbb{R}) \tag{6}$$

Here $g: V \to \mathbb{R}$ is fixed uniquely once σ is fixed. However, σ may vary up to isomorphism; therefore, a generic choice of σ is given by:

$$\sigma_{\bullet} = 0, \quad \sigma_{\star} = k \, \mathbb{1}_{\mathbb{R}}, \quad k \in \mathbb{R}$$
 (7)

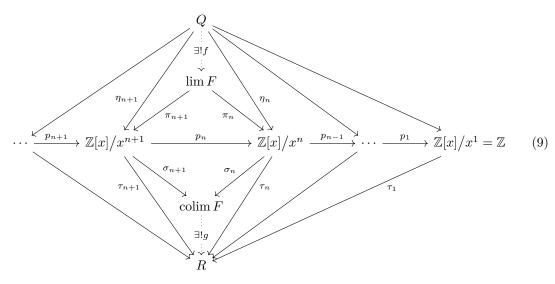
For such g, by the same arguments in (a), we have:

$$\exists ! \ g = \frac{1}{k} \eta_{\star}, \ \tau = \Delta(g) = \frac{1}{k} \Delta(\eta_{\star}), \quad s. t. \quad (2), (3) \ commutes$$
 (8)

Note that $k \in \mathbb{R}$, for every k there is a different g and τ ; hence there are $||\{k\}|| = ||\mathbb{R}||$ many choices of τ to make the diagram commute. In particular, for k = 1 we recover $\tau = \Delta(\eta_*)$.

2 Limit and colimit of polynomial ring:

By definition,



Here $p_n : \mathbb{Z}[x]/x^{n+1} \to \mathbb{Z}[x]/x^n$ is the natural projection.

Intuitively, if such $\lim F$ exists, it shall be the "smallest" object that "contains" $\mathbb{Z}[x]/x^n$ when $n \to \infty$. Note that $\mathbb{Z}[x]/x^n$ is naturally a \mathbb{Z}^n vector space:

$$\mathbb{Z}[x]/x^n \ni \sum_{m=0}^{n-1} a_m x^m \sim (a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}^n$$
 (10)

While $n \to \infty$, this gives an ∞ -tuple which corresponds to the formal power series¹:

$$\mathbb{Z}[[x]] \ni \sum_{m=0}^{\infty} a_m x^m \tag{11}$$

The difference between $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ is that the latter may contain infinite series while the former may not. Now we confirm that, indeed, $\lim F = \mathbb{Z}[[x]]$, along with natural projections $\pi_n \colon \mathbb{Z}[[x]] \to \mathbb{Z}[x]/x^n$.

In fact, the $f \colon R \to \lim F$ in (9) can be explicitly written down as:

$$f = \eta_1 + x \left(\frac{d}{dx} \eta_2\right) + x^2 \left(\frac{1}{2} \frac{d}{dx} \eta_3\right) + \dots = \sum_{m=0}^{\infty} x^m \left\{\frac{1}{m!} \frac{d^m}{dx^m} \eta_{m+1}\right\}$$

$$\sim (\eta_{1,0}, \eta_{2,1}, \dots, \eta_{n,n-1}, \dots)$$
(12)

¹ See Wikipedia: Formal power series. This is in fact the adic completion of $\mathbb{Z}[x]$. I would like to thank 刘逸华 & 谢 贤进 for this hint.

Here $\frac{1}{m!} \frac{d^m}{dx^m}$ is used to extract the a_{n-1} coefficient in $\mathbb{Z}[x]/x^n$; this is the last component of η_n , denoted by $\eta_{n,n-1}$. Any other choice of f will break communitativity of (9), hence f is fixed uniquely by $\mathbb{Z}[[x]]$ and π_n 's. Therefore, $\lim F = \mathbb{Z}[[x]]$.

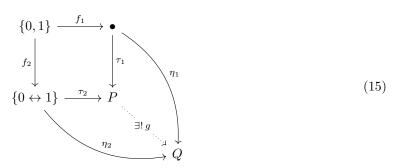
On the other hand, colim F is the "largest" object that any map out of $\mathbb{Z}[x]/x^n$ must "passes through". Also, this should hold for all $n \in \mathbb{Z}_+$. Naturally, projections $\sigma_n \colon \mathbb{Z}[x]/x^n \to \mathbb{Z}$ satisfy the above requirements; we have:

$$\sigma_n \colon x \longmapsto 0, \quad \sum_{m=0}^{n-1} a_m x^m \longmapsto a_0,$$
 (13)

$$\sigma_{n+1} = p_1 \circ p_2 \circ \cdots p_n \tag{14}$$

 $g: \mathbb{Z} \to R$ in (9) is fixed uniquely for such choice of σ_n ; in fact, descend along the p_n tower in (9), and we have: $\tau_{n+1} = \tau_n \circ p_n = \tau_{n-1} \circ p_{n-1} \circ p_n = \cdots = \tau_1 \circ p_1 \circ p_2 \circ \cdots \circ p_n = \tau_1 \circ \sigma_{n+1}, \ \forall \ n \in \mathbb{Z}_+,$ hence $\exists ! \ g = \tau_1$. Therefore, colim $F = \mathbb{Z}$.

3 Example of push-out in Groupoid:



Following the same observation as before, the push-out P is the "largest" object that any map out of \bullet and $\{0 \leftrightarrow 1\}$ must pass through. By such universal property, P can be no larger than the coproduct: $\{\bullet\} \coprod \{0 \leftrightarrow 1\}$. However, we should also consider the equivalence imposed by:

$$\bullet \xleftarrow{f_1} \{0,1\} \xrightarrow{f_2} \{0 \leftrightarrow 1\} \tag{16}$$

Therefore, we simply have $P = \bullet$, with $\tau_{1,2}$ the natural projection. This can be verified with ease: we have $g = \eta_1$. It is unique since its image is a single point (with identity map to itself) $\star \in Q$, and the point \star is fixed by commutativity.

4 Product and coproduct in Ab:

For $G_{\alpha} \in \mathbf{Ab} \subset \mathbf{Group}$, note that we have:

$$Free : \mathbf{Set} \Longrightarrow \mathbf{Group} : Forget$$
 (17)

Therefore, for F: some diagram in $\underline{\mathbf{Group}}$, $\left[\lim \left(\operatorname{Forget} \circ F\right) = \operatorname{Forget} \circ \lim F\right]$ if $\lim F$ exists.

By definition, the product $\prod_{\alpha} G_{\alpha} \in \underline{\mathbf{Group}}$ is a limit, hence it is idential (as in $\underline{\mathbf{Set}}$) to the *direct product*, with additional entry-wise group multiplication. Same applies for the full subcategory: abelian group $\underline{\mathbf{Ab}} \subset \mathbf{Group}$.

On the other hand, the disjoint union of G_{α} 's as sets will not necessary be a group, the identities $\mathbb{1}_{\alpha} \in G_{\alpha}$ must be glued together to produce a group structure. Furthermore, free-forgetful adjunction

(17) implies that for F': some diagram in **Set**,

$$\operatorname{colim}\left(\operatorname{Free}\circ F'\right) = \operatorname{Free}\circ\operatorname{colim} F',\tag{18}$$

Whenever colim F' exists; in our case, colim F' is the disjoint union of sets: $\coprod_{\alpha} \text{Forget}(G_{\alpha})$. Therefore, we should construct a free object in $\underline{\mathbf{Ab}}$.

Here we restrict our discussion to $\underline{\mathbf{Ab}}$, since the coproduct in $\underline{\mathbf{Ab}}$ is not the same as in $\underline{\mathbf{Group}}$ —the free product of abelian group is not necessary abelian. Hence, the coproduct in $\underline{\mathbf{Ab}}$ shall be:

$$\coprod_{\alpha} G_{\alpha} = \bigoplus_{\alpha} G_{\alpha}, \quad i_{\alpha} \colon G_{\alpha} \longleftrightarrow \bigoplus_{\alpha} G_{\alpha}$$
 (19)

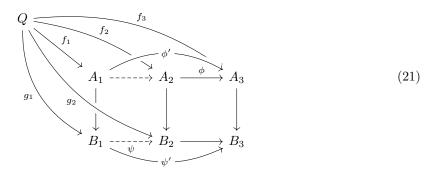
As a set, this is precisely the disjoint union with identities $0_{\alpha} \in G_{\alpha} \subset \underline{\mathbf{Ab}}$ glued together.

It is then straight-forward to verify its universal property: for $f_{\alpha} \colon G_{\alpha} \to H$,

$$\exists ! f \colon \bigoplus_{\alpha} G_{\alpha} \longrightarrow H, \quad (g_{\alpha})_{\alpha} \longmapsto \sum_{\alpha} f_{\alpha}(g_{\alpha})$$
 (20)

This is compatible with the abelian group multiplication. Note that for the summation to be well-defined, the coproduct must only contain finitely many components; otherwise it is identical to the product in $\underline{\mathbf{Ab}}$.

5 Composition of pull-backs:



- (a) If A_1 is the pull-back of A_2 and A_2 is the pull-back of A_3 , then given Q with f_3, g_1 , we have $g_2 = \psi \circ g_1$, and f_2 is fixed uniquely by universal property of A_2 , while f_1 is fixed uniquely by universal property of A_1 . Hence, A_1 is the pull-back of A_3 .
- (b) If A_1, A_2 are pull-backs of A_3 , then given Q with f_2, g_1 , we have $f_3 = \phi \circ f_2$, and f_2 is fixed uniquely by universal property of A_1 . Hence, A_1 is the pull-back of A_3 .

→ PAST WORK, AS TEMPLATE →

1 For $F_i \to E_i \xrightarrow{p_i} B$: coverings in $Cov_0(B)$ with E_i : connected and B: path connected and locally path connected, the following diagram commutes:

$$E_1 \xrightarrow{f} E_2$$
 $e_2 = f(e_1),$ $b = p_1(e_1) = p_2(e_2),$

To show that f is itself a covering, we need only verify that f is locally trivial with some discrete fiber F. In fact, given any $e_2 \in E_2$ and $b = p_2(e_2)$, there exists some neighborhood $U \subset B$ that the following diagram holds (by restriction):

$$U \times F_1 \xrightarrow{f} U \times F_2 \qquad e_1 = (b, k_1), \\ e_2 = (b, k_2(b, k_1)), \qquad k_i \in F_i$$

Generally, $k_2 = k_2(b, k_1)$ depends on the base point $b \in B$. However, since B is locally path connected, we can restrict U to be path connected, while $k_2 \in F_2$: discrete. Since continuous maps preserve path connectedness, k_2 is in fact independence of b, i.e. $k_2 = \varphi(k_1)$.

On the other hand, $\forall e_2 = (b, k_2) \in U \times \{k_2\} \subset E_2$, we have its preimage $f^{-1}(e_2) = \{b\} \times \varphi^{-1}(k_2)$. Note that E_2 is connected while $\varphi^{-1}(k_2) \in F_1$ is discrete; for the same reasoning as above, $\varphi^{-1}(k_2) = F$ is in fact independent of k_2 . This is the discrete fiber F we have been looking for. Hence f is also a covering map².

2 Cylinder with ends pinched — π_1 and universal cover:

$$Y = (X \times I)/(X \times \partial I) , \quad I = [0, 1]$$
(22)

Note that Y is homeomorphic to two cones³ $CX_1 \coprod CX_2$ with "bases" $X_i \subset CX_i$ and "vertices" v_i respectively identified: $X_1 \sim X_2$, $v_1 \sim v_2 \equiv v$. X is path connected and so is Y, hence we are free to choose $\pi_1(Y) = \pi_1(Y, y_0)$.

First note that paths that do *not* pass through the vertex v are all homotopic, since they are contained in a cone and cones are contractible⁴. Therefore all contributions to $\pi_1(Y)$ are loop classes that do pass through the vertex v. In other words, morphisms in $\Pi_1 Y$ are in one-to-one correspondence with morphisms in:

$$\Pi_1([0,1]/_{0\sim 1}) = \Pi_1 S^1 \tag{23}$$

Therefore,
$$\pi_1(Y) \cong \pi_1(S^1) = \mathbb{Z}$$
.

The universal cover \tilde{Y} of Y can be constructed by assigning an induced topology to the space of path classes, same as in the general proof of its existence. Since Y is "degenerate" at its vertex,

 $^{^2}$ Reference: math.stackexchange.com/a/109774.

 $^{^3~}$ See discussions from Problem Set $N\!\!\!_{2}1.$

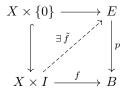
⁴ $[\gamma_1] = [\gamma_2 \star \gamma_2^{-1} \star \gamma_1] = [\gamma_2].$

this is equivalent to "cutting open" Y at its vertex v, and joining \mathbb{Z} copies them end-to-end. More explicitly, it can be written as:

$$\tilde{Y} = (X \times \mathbb{R}) / \sim, \quad (x, n) \sim (x', n), \ \forall \ x \in X, \ n \in \mathbb{Z}$$
 (24)

While the covering map: $\tilde{Y} \ni [x, t] \mapsto [x, t - \lfloor t \rfloor] \in Y$, here $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$.

$\boxed{\mathbf{3}}$ π_1 of fiber in fibration:



For $F \to E \xrightarrow{p} B$: fibration, by homotopy lifting property (HLP), any homotopy in B can be uniquely lifted to path class in E, provided some "initial condition" $X \times \{0\}$. This leads to the following results:

(a) For B: simply-connected, take any loop class $[\tilde{\gamma}] \in \pi_1(E, e)$ as initial condition; its projection $[p \circ \tilde{\gamma}] \in \pi_1(B, b) = \{[\mathbbm{1}_b]\}$ is trivial, i.e. $p \circ \tilde{\gamma} \simeq \mathbbm{1}_b$. By HLP, such homotopy can be lifted into E, i.e.

$$p \circ \tilde{\gamma} \simeq \mathbb{1}_b \quad \xrightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}' = \mathbb{1}_b$$
 (25)

In other words, $\tilde{\gamma} \simeq \tilde{\gamma}' \subset p^{-1}(b)$, i.e. any loop in E is homotopic to some loop in $p^{-1}(b) \cong F$. This implies a surjective group homomorphism $\pi_1(p^{-1}(b), e) \to \pi_1(E, e)$, i.e. an epimorphism. \square

(b) For E: simply-connected, take any loop class $[\gamma] \in \pi_1(B, b)$ and consider its lifting $[\tilde{\gamma}]$. Note that in general $\tilde{\gamma}$ is *not* a loop; however, we have $p \circ \tilde{\gamma} = \gamma$, hence $\tilde{\gamma}(0), \tilde{\gamma}(1) \in p^{-1}(b)$. In general, we have:

$$\gamma \simeq \gamma' \quad \xrightarrow{\text{lift}} \quad \tilde{\gamma} \simeq \tilde{\gamma}', \quad p \circ \tilde{\gamma}^{(\prime)} = \gamma^{(\prime)}$$
 (26)

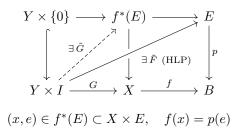
By continuity, $\tilde{\gamma}(0)$, $\tilde{\gamma}'(0) \in F_0$: a path component of $p^{-1}(b)$; similarly, $\tilde{\gamma}(1)$, $\tilde{\gamma}'(1) \in F_1$. In other words, the start and end points of $\tilde{\gamma}$ are confined in path components F_0 and F_1 , respectively. Hence a loop class in $\pi_1(B,b)$ maps to transport between path components:

$$T_{(\cdot)}(e) : \pi_1(B,b) \longrightarrow \pi_0(p^{-1}(b))$$

$$[\gamma] \longmapsto T_{[\gamma]}(e)$$
(27)

As a matter of fact, $T_{(\cdot)}(e)$ is a bijection. For $T_{[\gamma]} = T_{[\gamma']}$, they are characterized by two lifted paths $\tilde{\gamma}, \tilde{\gamma}'$; since E is simply connected, they are always homotopic: $\tilde{\gamma} \simeq \tilde{\gamma}'$, hence $[\gamma] = [\gamma']$ by projection p. This means that T is injective. Surjectivity also follows from projection $\gamma = p \circ \gamma'$. Therefore, $T_{(\cdot)}(e)$ gives a bijection between $\pi_1(B,b)$ and $\pi_0(p^{-1}(b))$.

4 Pull-back of fibration is fibration:



We need only verify that $f^*(E) \to X$ also has HLP, i.e. the existence of \tilde{F} in the above diagram⁵. By HLP of $E \xrightarrow{p} B$, $\exists \tilde{F} \colon Y \times I \to E$ as shown above. We can use \tilde{F} to construct \tilde{G} explicitly; in fact, first consider:

$$\tilde{G} \colon Y \times I \longrightarrow X \times E$$

$$(y,t) \longmapsto (G(y,t), \tilde{F}(y,t))$$
(28)

Note that $f \circ G = p \circ \tilde{F}$; compared with the definition of $f^*(E)$, this implies that the image of \tilde{G} lies within $f^*(E) \subset X \times E$, hence after restriction of its codomain, \tilde{G} becomes a well-defined lifting of G into $f^*(E)$. Therefore, $f^*(E) \to X$ has HLP, i.e. it is also a fibration.

5 More properties of fibration:

- (a) By HLP, given any initial condition $e \in p^{-1}(b_1)$, lifting of any path $b_1 \xrightarrow{\gamma} b_2$ exists. The lifted path with dependence of e can then be written as $F \colon p^{-1}(b_1) \times I \to E$. This is just a generalization of $\boxed{3}$ for non-loop paths.
- (b) Similarly, transport $T_{[\gamma]}$ defined in $\boxed{\mathbf{3}}$ can be generalized for non-loop paths. $T_{[\gamma]}$ is well-defined for path class $[\gamma]$, since by HLP homotopic paths can be lifted to homotopy in E. Therefore, the transport is fixed up to homotopy, i.e.

$$T: \operatorname{Hom}_{\Pi_{1}B}(b_{0}, b_{1}) \longrightarrow \operatorname{Hom}_{\underline{\operatorname{hTop}}} \left(p^{-1}(b_{0}), p^{-1}(b_{1}) \right)$$

$$[\gamma] \longmapsto T_{[\gamma]}$$

$$(29)$$

Note that T defined in this way is also independent of the choice of F, since F simply specifies the starting point of the lifted path; no matter which F we choose, the lifted paths will always be homotopic in E. Hence T is well-defined in the above sense.

- (c) T defined above is a functor: $\Pi_1 B \to \underline{\mathbf{hTop}}$. To verify this, we need only check that it is compatible with composition and maps identity morphisms to identity morphisms. Indeed, $T_{[\mathbbm{1}_b]} = [\mathbbm{1}_{p^{-1}(b)}]$, and $T_{[\gamma']\star[\gamma]} = T_{[\gamma'\star\gamma]} = T_{[\gamma']} \circ T_{[\gamma]}$ by joining two lifted paths (up to homotopy). \square
- (d) For B: path connected, there exists an isomorphism between any two objects in $\Pi_1 B$ (a path connecting any two points in B), which is mapped to isomorphisms between fibers $p^{-1}(b)$ in **hTop**. Hence any two fibers of $E \xrightarrow{p} B$ have the same homotopy type.

Notice that $f^*(E)$ is the limit of the diagram, hence this is automatically true by the universal property of $f^*(E)$. I would like to thank 刘逸华 for pointing this out. For now, we will stick to a more traditional proof.