Algebraic Topology №1

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1 **Equivalence of categories is fully faithful**:

$F\colon \mathcal{C} \to \mathcal{D}$ equivalence of categories, i.e. $\exists\, G\colon \mathcal{D} \to \mathcal{C}$, s.t.

$$G \circ F \simeq \mathbb{1}_{\mathcal{C}}, \quad F \circ G \simeq \mathbb{1}_{\mathcal{D}} \tag{1}$$

Here "$\simeq$" means naturally isomorphic as functors, i.e.,

$$\exists\, \tau\colon G \circ F \Rightarrow \mathbb{1}_{\mathcal{C}}, \quad \sigma\colon F \circ G \Rightarrow \mathbb{1}_{\mathcal{D}} : \text{ natural isomorphisms} \tag{2}$$

By the definition of natural transformation, for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have:

$$\begin{array}{ccc} G \circ F(A) & \xrightarrow{G \circ F(f)} & G \circ F(B) \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

$$\tau_B \circ (G \circ F)(f) \circ \tau_A^{-1} = f, \quad \forall\, f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \tag{3}$$

Here $\tau_{A,B}$ are isomorphisms, which means that $G \circ F$ must be a bijection between hom-sets, which further implies that F is injective and G is surjective. Switch the roles of F, G, we find that G is injective and F is surjective. Therefore, F, G are both fully faithful. ■

2 **Forgetful functors to Set are often representable**:

For $F\colon$ **Group** $\to$ **Set**, consider the free group generated by a single element $\mathbb{Z}$. We have:

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}, -)\colon \ \underline{\mathbf{Group}} &\longrightarrow \underline{\mathbf{Set}} \\ G &\longmapsto \mathrm{Hom}(\mathbb{Z}, G) \end{aligned} \tag{4}$$

This is a covariant functor representable by $\mathbb{Z}$.

On the other hand, $\mathrm{Hom}(\mathbb{Z}, G)$ consists of group homomorphisms:

$$\mathrm{Hom}(\mathbb{Z}, G) = \left\{ \begin{matrix} \mathbb{Z} \to G \\ 1 \mapsto g \end{matrix} \,\middle|\, g \in G \right\} \tag{5}$$

More specifically, to fix any $\mathbb{Z} \to G$, we need only assign its generator[1] $1 \mapsto g$. Image of any other $\mathbb{Z}$ element is generated automatically from the group law, without further specifications. This means that the hom-set is in one-to-one correspondence with G elements (as a set). Therefore, $F \cong \mathrm{Hom}_{\underline{\mathbf{Group}}}(\mathbb{Z}, -)$, i.e. forgetful $F\colon$ **Group** $\to$ **Set** is representable by $\mathbb{Z}$. □

Similarly, for $F\colon$ **Ring** $\to$ **Set**, the free object generated by some generic element x is $\mathbb{Z}[x]$, the polynomial ring in one variable; we have:

$$F \cong \mathrm{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], -), \quad \mathrm{Hom}_{\underline{\mathbf{Ring}}}(\mathbb{Z}[x], R) = \left\{ \begin{matrix} \mathbb{Z}[x] \to R \\ x \mapsto r \end{matrix} \,\middle|\, r \in R \right\} \tag{6}$$

Lesson: Forgetful **Cat** $\to$ **Set** are often representable by the free object in **Cat**. ■

[1] Note that $0 \in \mathbb{Z}$ is the group identity of addiction group $\mathbb{Z}$, not $1 \in \mathbb{Z}$.

3 Properties of contractible space:

(a) X contractible: $\mathbb{1}_X \simeq f_0\colon X \to X$ some constant map, $f_0(X) = \{x_0\}$. We can restrict the codomain of f_0 so that $f_0\colon X \to \{x_0\}$, in this way we have:

$$X \xrightarrow{f_0} \{x_0\} \hookrightarrow X \simeq \mathbb{1}_X, \tag{7.1}$$

$$\{x_0\} \hookrightarrow X \xrightarrow{f_0} \{x_0\} \simeq \mathbb{1}_{\{x_0\}}, \tag{7.2}$$

This means that $f_0\colon X \to \{x_0\}$ isomorphic in $\underline{\mathbf{hTop}} = \underline{\mathbf{Top}}/\simeq$, which is precisely the definition of homotopic equivalence $X \simeq \{x_0\}$. $(\Rightarrow)$

On the other hand $(\Leftarrow)$, if $X \simeq \{x_0\}$, there exists some $f_0\colon X \to \{x_0\}$ that fulfills (7). We can then extend the codomain s.t. $f_0\colon X \to X$, in this way (7.1) reads $f_0 \simeq \mathbb{1}_X$, i.e. X is contractible. Therefore, X contractible iff. homotopic equivalent to a single point. $\blacksquare_{(a)}$

(b) $\forall\ X$: Topological space, we can define its *cone* as[2]:

$$CX = (X \times I)/(X \times \{0\}), \quad I = [0,1] \tag{8}$$

i.e. gluing together one end of the cylinder $X \times I$. Naturally $X \subset CX$ as a subspace; now we show that CX is contractible. Using (a), we need only show that $\mathbb{1}_{CX} \simeq f_0$ some constant map.

In fact, any point in CX can be uniquely labeled by $[x,h] \in X \times I$, with the exception of the vertex $v \sim [x,0] \sim [x',0]$, $\forall\ x, x' \in X$. We can then construct a homotopy F by shrinking the cone towards the vertex v:

$$\begin{gathered} F\colon CX \times I \to CX, \quad F\big([x,h],t\big) = [x, h\cdot t], \\ F|_{CX\times 0} = v = \text{const}, \quad F|_{CX\times 1} = \mathbb{1}_X \end{gathered} \tag{9}$$

This confirms that $\mathbb{1}_{CX} \simeq v$: constant map. By (a), CX is contractible. $\blacksquare_{(b)}$

(c) For $Y \simeq \{y_0\}$ contractible, given any $g\colon X \to Y$, we can deform the image $g(X) \subset Y$ to a single point, hence $g \simeq y_0$: constant map. More precisely, we have:

$$\exists\ G\colon X \times I \to Y, \quad \text{s.t.} \quad G|_{X\times 0} = y_0 = \text{const}, \quad G|_{X\times 1} = g \tag{10}$$

Such G can be explicitly constructed using $\mathbb{1}_Y \simeq y_0$:

$$F\colon Y \times I \to Y, \quad F|_{Y\times 0} = y_0 = \text{const}, \quad F|_{Y\times 1} = \mathbb{1}_Y, \tag{11}$$

$$G(x,t) = F\big(g(x),t\big) \tag{12}$$

In summary, we have proven that $g \simeq y_0$, $\forall\ g \in \mathrm{Hom}_{\underline{\mathbf{Top}}}(X,Y)$. By definition, this means that $\mathrm{Hom}_{\underline{\mathbf{hTop}}}(X,Y) = \mathrm{Hom}_{\underline{\mathbf{Top}}}(X,Y)\big/_{\simeq} = \big\{[y_0]\big\}$ a single point. $\blacksquare_{(c)}$

(d) For $X \simeq \{x_0\}$ contractible, similar to (11), we have homotopy $F\colon X \times I \to X$. Given any $f\colon X \to Y$, the composition $f \circ F\colon X \times I \to Y$ yields $f \simeq f(x_0)$: constant map.

[2] See `Wikipedia`: *Cone (topology)*.

Furthermore, for Y: path connected, there is a path $\gamma\colon I \to Y$ connecting $f(x_0)$ and some $y_0 \in Y$, therefore $f(x_0) \simeq y_0\colon X \to Y$ constant maps. More precisely, we have:

$$\gamma\colon I \to Y, \quad \gamma(0) = y_0, \quad \gamma(1) = f(x_0), G\colon X \times I \to Y, \quad G(x,t) = \gamma(t) \tag{13}$$

Which gives $f(x_0) \simeq y_0$, $\forall f$, independent of the choice of f. This means that $f \simeq f(x_0) \simeq y_0$: constant map, therefore $\mathrm{Hom}_{\underline{\mathbf{hTop}}}(X,Y) = \big\{[y_0]\big\}$ a single point. $\blacksquare_{(d)}$

4 Example of homotopic inequivalence[3]:

$$X = \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\}, \quad Y = \{0\} \cup \mathbb{Z}_+ \\ X, Y \subset \mathbb{R}\colon \text{subspace topology} \tag{14}$$

Assume $X \simeq Y$, then similar to (7), we have $Y \xrightarrow{g} X \xrightarrow{f} Y \simeq \mathbb{1}_Y$. However, note that Y has discrete topology, in such case any map $f \circ g$ homotopic to $\mathbb{1}_Y$ must be $\mathbb{1}_Y$ itself: $f \circ g = \mathbb{1}_Y$.

More specifically, consider:

$$F\colon Y \times I \to Y, \quad F|_{Y\times 0} = f \circ g, \quad F|_{Y \times 1} = \mathbb{1}_Y \tag{15}$$

Any point $n \in Y$ is both open and closed, therefore its pre-image $F^{-1}(n) \subset Y \times I$ is also both open and closed, and by $F|_{Y\times 1} = \mathbb{1}_Y$ we know that $F(y,1) = y$, $(y,1) \in F^{-1}(y)$, therefore the only possibility is that $F(\{y\} \times I) = y$, i.e. $f \circ g = \mathbb{1}_Y$, which implies that g is injective and f is surjective.

However, $f\colon X \to Y$ cannot be surjective due to the complication around $0 \in X$. Consider $f^{-1}\big(f(0)\big) \ni 0$, since $f(0) \in Y$ both open and closed, $f^{-1}\big(f(0)\big) \subset X$ must also be both open and closed. But any open set $U \subset X$ is induced via subspace topology $X \subset \mathbb{R}$; for $0 \in U \subset X \subset \mathbb{R}$, U must contain ∞-many elements:

$$\left\{ \frac{1}{n} \,\middle|\, n \geq N_0 \right\} \subset U \subset f^{-1}\big(f(0)\big), \quad \text{for some } N_0, \text{ for any } U \ni x \tag{16}$$

Hence $f(X) = f(0) \cup f(\{\frac{1}{n} \,|\, n < N_0\})$, $f(X) \subset Y$ a finite set, i.e. $f\colon X \to Y$ is never surjective. Therefore, $X \not\simeq Y$ by contradiction. $\blacksquare$

[3] This proof is produced thanks to helpful insights from 谷夏 and 於子雄.

5 Fundamental group of topological group is abelian[4]:

From a categorical point of view, the fundamental group $\pi_1(G)$ of a topological group G can be seen as a functor:

$$G \in \underline{\textbf{TopGroup}} \hookrightarrow \underline{\textbf{Top}} \xrightarrow{\pi_1} \underline{\textbf{Group}} \ni \pi_1(G) \tag{17}$$

$\underline{\textbf{TopGroup}} \subset \underline{\textbf{Top}}$ is a subcategory with additional group structure, i.e. $(G, \cdot) \in \underline{\textbf{TopGroup}}$ is a *group object*[5] in $\underline{\textbf{Top}}$, with "$\cdot$" denoting its product operation $(\cdot)\colon G \times G \to G$. Correspondingly, $\pi_1(\underline{\textbf{TopGroup}})$ should be *group objects of* $\underline{\textbf{Group}}$, which have an *additional* group structure $(\star) = \pi_1(\cdot)$, along with the usual group product "$*$" in $\underline{\textbf{Group}}$.

In total, we have three different group structures (represented by their product operation):

$$(\cdot)\colon \quad G \times G \to G, \tag{18}$$

$$(*)\colon \quad \pi_1(G) \times \pi_1(G) \to \pi_1(G), \tag{19}$$

$$(\star) = \pi_1(\cdot)\colon \quad \pi_1(G) \times \pi_1(G) \to \pi_1(G), \tag{20}$$

Note that $\pi_1(G) = \mathrm{Aut}_{\Pi_1(G)} \mathbb{1}_G$, i.e. loop classes $[\gamma]$ in G; $(*)$ is defined as joining two loops, while $(\star) = \pi_1(\cdot)$ is defined as the translation of loop classes by pointwise group product $(\cdot)$,

$$[\gamma_1] \star [\gamma_2] = [\gamma_1 \cdot \gamma_2] \tag{21}$$

With the above definitions, we observe that:

$$([\gamma_1] \star [\gamma_2]) * ([\eta_1] \star [\eta_2]) = ([\gamma_1] * [\eta_1]) \star ([\gamma_2] * [\eta_2]) \tag{22}$$

By definition, they are both equal to $[(\gamma_1 \cdot \gamma_2) * (\eta_1 \cdot \eta_2)]$. What's surprising is that by using only the group axioms and "distributive law" (22), we can show that $(\star)$ and $(*)$ must always coincide: $(\star) = (*)$, and they have to be in fact, commutative. This is the *Eckmann–Hilton argument*[6].

Proof of this argument is straight-forward; first, observe that the units of the two operations coincide:

$$1_\star = 1_\star \star 1_\star = (1_* * 1_\star) \star (1_\star * 1_*) \overset{(22)}{=\!=} (1_* \star 1_\star) * (1_\star \star 1_*) = 1_* * 1_* = 1_* \tag{23}$$

Further manipulation using (22) confirms that the two operations coincide and are commutative:

$$\begin{aligned} [\gamma] * [\eta] &= (1 \star [\gamma]) * ([\eta] \star 1) \overset{(22)}{=\!=} (1 * [\eta]) \star ([\gamma] * 1) \\ &= [\eta] \star [\gamma] \\ &= ([\eta] * 1) \star (1 * [\gamma]) \overset{(22)}{=\!=} ([\eta] \star 1) * (1 \star [\gamma]) \\ &= [\eta] * [\gamma] \end{aligned} \tag{24}$$

In summary, we find that the group objects in $\underline{\textbf{Group}}$ are indeed abelian groups, which means that $\pi_1(G)$ for $G \in \underline{\textbf{TopGroup}}$ must be abelian. ■

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[4] This proof is produced with the help of `math.stackexchange.com/q/727999`. Another (easier) proof lies in the fact that group translation induces π_1 conjugation, therefore $\gamma^{-1}\alpha\gamma = \alpha$, hence abelian.

[5] See `Wikipedia:` *Group object*.

[6] See `Wikipedia:` *Eckmann–Hilton argument*.